

Global relaxation of bistable solutions for gradient systems in one unbounded spatial dimension

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This paper is concerned with spatially extended gradient systems of the form

$$u_t = -\nabla V(u) + \mathcal{D}u_{xx},$$

where spatial domain is the whole real line, state-parameter u is multidimensional, \mathcal{D} denotes a fixed diffusion matrix, and the potential V is coercive at infinity. *Bistable* solutions, that is solutions close at both ends of space to stable homogeneous equilibria, are considered. For a solution of this kind, it is proved that, if the homogeneous equilibria approached at both ends belong to the same level set of the potential and if an appropriate (localized in space) energy remains bounded from below when time increases, then the solution approaches, when time approaches infinity, a pattern of stationary solutions homoclinic or heteroclinic to homogeneous equilibria. This result provides a step towards a complete description of the global behaviour of all bistable solutions that is pursued in a companion paper. Some consequences are derived, and applications to some examples are given.

1 Introduction

This paper deals with the global dynamics of nonlinear parabolic systems of the form

$$(1) \quad u_t = -\nabla V(u) + \mathcal{D}u_{xx},$$

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where time variable t and space variable x are real, spatial domain is the whole real line, the function $(x, t) \mapsto u(x, t)$ takes its values in \mathbb{R}^n with n a positive integer, \mathcal{D} is a fixed $n \times n$ positive definite symmetric real matrix, and the nonlinearity is the gradient of a scalar *potential* function $V : \mathbb{R}^n \rightarrow \mathbb{R}$, which is assumed to be regular (of class at least \mathcal{C}^2) and coercive at infinity (see hypothesis (H_{coerc}) in subsection 2.2 on page 5).

The main feature of system (1) is that it can be recast, at least formally, as the gradient flow of an energy functional. If (v, w) is a pair of vectors of \mathbb{R}^n , let $v \cdot w$ and $|v| = \sqrt{v \cdot v}$ denote the usual Euclidean scalar product and the usual Euclidean norm, and let

$$\langle v, w \rangle_{\mathcal{D}} = v \cdot \mathcal{D}w \quad \text{and} \quad |v|_{\mathcal{D}} = \sqrt{\langle v, v \rangle_{\mathcal{D}}}$$

denote the scalar product associated to \mathcal{D} and the corresponding norm, respectively. If $(x, t) \mapsto u(x, t)$ is a solution of (1), the *energy* (or *Lagrangian* or *action*) functional of the solution reads:

$$(2) \quad \mathcal{E}[u(\cdot, t)] = \mathcal{E}[x \mapsto u(x, t)] = \int_{\mathbb{R}} \left(\frac{|u_x(x, t)|_{\mathcal{D}}^2}{2} + V(u(x, t)) \right) dx.$$

Its time derivative reads, at least formally,

$$(3) \quad \frac{d}{dt} \mathcal{E}[u(\cdot, t)] = - \int_{\mathbb{R}} |u_t(x, t)|^2 dx \leq 0,$$

and system (1) can formally be rewritten as:

$$u_t(\cdot, t) = - \frac{\delta}{\delta u} \mathcal{E}[u(\cdot, t)].$$

If system (1) is considered on a *bounded* spatial domain with boundary conditions that preserve this gradient structure, then the integrals in (2) and (3) converge, thus the system is really — and not only formally — of gradient type. In this case the dynamics is (at least from a qualitative point of view) fairly well understood, up to a fine description of the global attractor that is compact and made of the unstable manifolds of stationary solutions [15, 32]. According to LaSalle's principle, every solution approaches the set of stationary solutions (and even a single stationary solution if the potential is analytic, [30]).

If space is the whole real line and the solutions under consideration are only assumed to be bounded, then the gradient structure above is only formal and allows a much richer phenomenology (the full attractor is far from being fully understood in this case, see the introduction of [14] and references therein). A salient feature is the occurrence of travelling fronts, that is travelling waves connecting homogeneous equilibria at both ends of space. Those solutions are known to play a major role in the asymptotic behaviour of “many” initial conditions; roughly speaking they are two classes of them, depending on the nature of the invaded equilibrium: monostable fronts, where an unstable equilibrium is replaced by a stable one, and bistable fronts, where the invaded equilibrium also is stable. A reasonably wide class of solutions, sufficiently large to capture the convergence to travelling fronts while limiting the complexity of the dynamics encountered is made

of solutions that are close to homogeneous equilibria at both ends of space, at least for large times. And among such solutions the simplest case is that of *bistable* solutions, when those equilibria at both ends of space are stable.

In the late seventies, substantial breakthroughs have been achieved by P. C. Fife and J. B. McLeod about the global behaviour of such *bistable* solutions in the *scalar* case (n equals 1). Their results comprise global convergence towards a bistable front [8], global convergence towards a “stacked family of bistable fronts” [9], and finally, in the case of a bistable potential, a rather complete description of the global asymptotic behaviour of all solutions that are sufficiently close, at infinity in space, to the local (non global) minimum point [10].

The aim of this paper, together with the companion papers [13, 26, 27], is to make a step further in this program, by extending those results to the case of *systems*, and by providing for such systems a complete description of the asymptotic behaviour of all bistable solutions (under generic hypotheses on the potential V). Concerning the nature of the arguments involved in the proofs, the main difference with respect to Fife and McLeod’s approach is the fact that the maximum principle does not hold any more for systems. It turns out, though, that a purely variational approach is sufficient to recover the results obtained by these authors, exploiting the fact that a gradient structure similar to the one above exists in every travelling referential (though only in the case where the diffusion matrix \mathcal{D} is the identity matrix, unfortunately). Observe by the way that this gradient structure in every travelling referential was already mentioned and used by Fife and McLeod in their seminal (initial) paper [8] of 1977 (see p. 350).

Roughly speaking, the global behaviour of every bistable solution is as follows (see [27] for more details): each of the two spatially homogeneous equilibria at the ends of space may (or not) be invaded by a bistable travelling front, which may itself be followed by a second one (at a speed that is not larger than the one of the first front), and so on. Each of these travelling fronts replaces a (local) minimum point of the potential by another where the value of the potential is lower. Since the potential is bounded from below, the number of fronts in these two “stacked families” (one at each end of space) is finite (and by the way possibly zero). The two equilibria left behind the “last” front of each of these two families must belong to the same level set of the potential, and behind these “last” fronts, the solution relaxes towards the set of stationary solutions that are homoclinic or heteroclinic to critical points — they will be assumed to be (local) minimum points — in this level set.

The purpose of this paper is to treat (only) the “relaxation” part of this program. To be more explicit, it is to describe the asymptotic behaviour of bistable solutions connecting (local) minimum points in the same level set of the potential and having a (properly localized) energy that remains bounded from below (that is, for which the equilibria at both ends of space are *not* “invaded” by travelling fronts). As a consequence, the gradient structure in travelling frames will not be required.

There is a huge amount of literature about relaxation of solutions for systems like (1). A tremendous work was achieved to obtain precise quantitative information about the approach to stationary solutions and the metastable dynamics (“dormant instability”) resulting from the long range interaction between these (spatially localized) stationary

solutions. Often, this has been done on the simplest possible models exhibiting these phenomena (like the Allen-Cahn equation), often in the scalar case n equals 1, with the use of the maximum principle, sometimes on a bounded domain (and also by the way often in higher space dimension), often for a potential taking only nonnegative values, and often for solutions of finite energy. The papers of S.-I. Ei [7] and F. Béthuel, G. Orlandi, and D. Smets [4] — both especially relevant with respect to the present work — contain a more complete list of references together with short historical reviews.

The purpose of this paper is more modest, since the results that will be proved are purely qualitative (they only concern the asymptotic dynamics after an arbitrarily long interval of time for which no quantitative estimate will be given). On the other hand, the hypotheses we will have to deal with (in relation with the purpose of describing in [27] the global dynamics of all bistable solutions) are slightly more general than those usually made in the literature mentioned above. Besides the fact that no maximum principle is available for systems and that the set of stationary solutions is a priori unknown, we must consider potentials that may take negative values and solutions for which expression (2) of energy (integral on the whole real line) may be infinite. The difficulties to overcome are thus to control the behaviour of bistable solutions at both ends of space (in a way sufficient to ensure an approximate decrease of a localized energy), and to prove convergence towards the set of stationary solutions that are homoclinic or heteroclinic to homogeneous equilibria without any a priori information about this set. The results (Theorems 1 and 2 below) are eventually nothing but a purely qualitative (thus weaker) version of many well-known results on more specific examples or with more specific hypotheses, especially if compared with the results proved by Béthuel, Orlandi, and Smets in [4].

2 Assumptions, notation, and statement of the results

2.1 Local semi-flow in uniformly local Sobolev space

Let us denote by X the uniformly local Sobolev space $H_{\text{ul}}^1(\mathbb{R}, \mathbb{R}^n)$ (its definition is recalled in subsection 3.1). This space is the most convenient with respect to estimates on the localized energy and localized L^2 -norm of the solutions that are used along the paper. However, due to the smoothing properties of system (1), the choice of the functional framework is not crucial, and every statement remains true if X denotes, instead of $H_{\text{ul}}^1(\mathbb{R}, \mathbb{R}^n)$, the more familiar Banach space $\mathcal{C}_b^1(\mathbb{R}, \mathbb{R}^n)$ of functions of class \mathcal{C}^1 that are uniformly bounded together with their first derivative. Accordingly, it is within the functional framework $X = H_{\text{ul}}^1(\mathbb{R}, \mathbb{R}^n)$ that the statements are the least sensitive to regularization properties, and thus most appropriate to further generalizations to a wider class of systems, for instance hyperbolic systems (see sub-subsection 2.14.5).

System (1) defines a local semi-flow in X (see for instance D. B. Henry's book [16]).

2.2 Coercivity of the potential and global semi-flow

Let us assume that the potential function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is of class \mathcal{C}^k with k not smaller than 2 (see subsection 3.2), and is strictly coercive at infinity in the following sense:

$$(\mathbf{H}_{\text{coerc}}) \quad \lim_{R \rightarrow +\infty} \inf_{|u| \geq R} \frac{u \cdot \nabla V(u)}{|u|^2} > 0$$

(or in other words there exists a positive quantity ε such that the quantity $u \cdot \nabla V(u)$ is larger than $\varepsilon|u|^2$ as soon as $|u|$ is sufficiently large).

According to this hypothesis ($\mathbf{H}_{\text{coerc}}$), the semi-flow of system (1) is actually global, in other words solutions are defined up to $+\infty$ in time (details are given in subsection 3.2). Let us denote by $(S_t)_{t \geq 0}$ this semi-flow. Then, for every u_0 in X , the solution corresponding to the initial data u_0 reads: $(x, t) \mapsto (S_t u_0)(x)$ and is defined for all x in \mathbb{R} and t in $[0, +\infty)$.

2.3 Bistable solutions: definition and notation

Our targets are bistable solutions, let us define them formally. In the definition below and everywhere in this paper, the term “minimum point” denotes a point where a function — namely the potential V — reaches a local *or* global minimum, and the adjective “nondegenerate” means (for a minimum point) that the Hessian matrix of the function at this point is positive definite.

Definition. A solution $(x, t) \mapsto u(x, t)$ of system (1) is called a *bistable solution* if there are two (possibly equal) nondegenerate minimum points m_- and m_+ of the potential V such that the quantities:

$$\limsup_{x \rightarrow -\infty} |u(x, t) - m_-| \quad \text{and} \quad \limsup_{x \rightarrow +\infty} |u(x, t) - m_+|$$

both approach 0 when time approaches $+\infty$. More precisely, such a solution is called a *bistable solution connecting m_- to m_+* (see figure 1). A function u_0 in X is called a

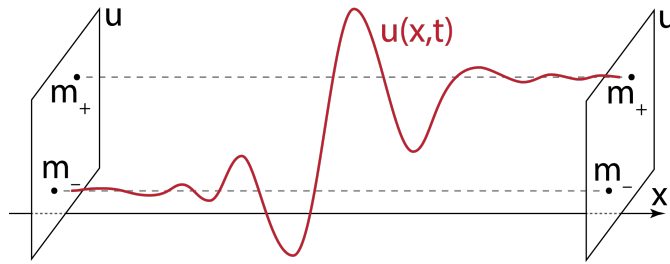


Figure 1: A bistable solution connecting m_- to m_+ .

bistable initial condition (connecting m_- to m_+) if the solution of system (1) corresponding to this initial condition is a bistable solution (connecting m_- to m_+).

Let m_- and m_+ denote two nondegenerate minimum points (possibly equal) of the potential V .

Notation. Let

$$X_{\text{bist}}(m_-, m_+)$$

denote the subset of X made of bistable initial conditions connecting m_- to m_+ .

By construction, this set is positively invariant under the semi-flow of system (1). It will be proved in section 4 (Corollary 1) that it is nonempty, open in X (for the usual norm on this function space), and that it contains all functions sufficiently close to the minimum points m_- and m_+ at the ends of space.

2.4 Level set zero of the potential: notation and hypotheses

In this paper, only bistable solutions connecting minimum points in the same level set of V will be considered. For convenience, we will assume that this level set corresponds to the value 0. Thus the following notation will be used:

$$V^{-1}(\{0\}) = \{u \in \mathbb{R}^n : V(u) = 0\}.$$

and the following “normalization” hypothesis will be made “without loss of generality”:

(H_{norm}) There exists at least one nondegenerate minimum point of V in the level set $V^{-1}(\{0\})$. In other words, the set

$$\{u \in \mathbb{R}^n : V(u) = 0 \text{ and } \nabla V(u) = 0 \text{ and } D^2V(u) \text{ is positive definite}\}$$

is nonempty.

The following additional hypothesis will be required at various stages of the proofs. The question of whether this hypothesis is required for the validity of the results stated below will unfortunately not be answered in this paper (for the reason that the author does not know the answer, see comments in sub-subsection 2.14.6).

(H_{min}) All critical points of V in the level set $V^{-1}(\{0\})$ are nondegenerate minimum points. In other words, at every point of \mathbb{R}^n where both V and its gradient vanish, the Hessian matrix of V is positive definite.

Note that for a generic potential V satisfying (H_{coerc}) the critical points are nondegenerate thus their number is finite and they belong to distinct level sets of V , and as a consequence (H_{min}) follows from (H_{norm}). In practical examples it often occurs, however, that several critical points belong to the level set $V^{-1}(\{0\})$, but this follows in general from the existence of a symmetry for V (like in examples (a) and (b) of figure 2), so that these critical points have the same Morse index; thus in such cases again (H_{min}) follows from (H_{norm}).

2.5 Notation for the Hamiltonian system of stationary solutions

A stationary solution $x \mapsto u(x)$ of system (1) is a function from \mathbb{R} to \mathbb{R}^n that is solution of the second order differential system in \mathbb{R}^n :

$$(4) \quad \mathcal{D}u'' = \nabla V(u),$$

or equivalently of the first order differential system in \mathbb{R}^{2n} :

$$(5) \quad \begin{cases} du/dx = v \\ dv/dx = \mathcal{D}^{-1} \nabla V(u) \end{cases}$$

which is a Hamiltonian system. Indeed, if the Hamiltonian H and the nondegenerate skew-symmetric matrix Ω are defined as:

$$(6) \quad H : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}, \quad (u, v) \mapsto \frac{|v|_{\mathcal{D}}^2}{2} - V(u) \quad \text{and} \quad \Omega = \begin{pmatrix} 0 & \mathcal{D}^{-1} \\ -\mathcal{D}^{-1} & 0 \end{pmatrix}$$

then the system (5) can be rewritten as

$$\frac{d}{dx} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{D}^{-1} \\ -\mathcal{D}^{-1} & 0 \end{pmatrix} \begin{pmatrix} -\nabla V(u) \\ \mathcal{D}v \end{pmatrix} = \Omega \cdot \nabla H(u, v).$$

and the Hamiltonian is a conserved quantity for this system. The (formal) energy defined in (2) is the integral of the Lagrangian

$$L : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}, \quad (u, v) \mapsto \frac{|v|_{\mathcal{D}}^2}{2} + V(u).$$

2.6 Notation for bistable stationary solutions in Hamiltonian level set zero

The statement of the main results requires additional notation concerning the set of stationary solutions that will be approached.

Notation. Let \mathcal{M}_0 denote the set of nondegenerate minimum points of V in the level set $V^{-1}(\{0\})$ — according to hypothesis (H_{\min}) this set is not smaller than the set of critical points of V in the same level set $V^{-1}(\{0\})$:

$$\begin{aligned} \mathcal{M}_0 &= \{u \in V^{-1}(\{0\}) : \nabla V(u) = 0\} \\ &= \{u \in V^{-1}(\{0\}) : \nabla V(u) = 0 \text{ and } D^2V(u) \text{ is positive definite}\}. \end{aligned}$$

Observe by the way that hypotheses (H_{coerc}) and (H_{\min}) ensure that this set is finite.

Notation. Let \mathcal{S} denote the set of stationary solutions of system (1), that is of solutions $x \mapsto u(x)$ of the second order (Hamiltonian) system (4) defined on the whole real line.

If (m_-, m_+) is a pair of minimum points of V in the level set $V^{-1}(\{0\})$ (that is a pair of points of \mathcal{M}_0 , that might be equal or different), let

$$\mathcal{S}_{\text{bist}}(m_-, m_+)$$

denote the set of *bistable stationary solutions connecting m_- to m_+* , that is the set of functions $x \mapsto u(x)$ in \mathcal{S} satisfying:

$$u(x) \xrightarrow{x \rightarrow -\infty} m_- \quad \text{and} \quad u(x) \xrightarrow{x \rightarrow +\infty} m_+$$

(including the homogeneous solution $u \equiv m_+$ if $m_+ = m_-$). Obviously, the set $\mathcal{S}_{\text{bist}}(m_-, m_+)$ is exactly made of stationary solution of system (1) that are altogether bistable solutions connecting m_- to m_+ , in other words,

$$\mathcal{S}_{\text{bist}}(m_-, m_+) = \mathcal{S} \cap X_{\text{bist}}(m_-, m_+).$$

Let

$$\mathcal{S}_{\text{bist}}(\mathcal{M}_0)$$

denote the union, for all pairs (m_-, m_+) of minimum points of V in the level set $V^{-1}(\{0\})$, of the sets $\mathcal{S}_{\text{bist}}(m_-, m_+)$. With symbols:

$$\mathcal{S}_{\text{bist}}(\mathcal{M}_0) = \bigsqcup_{(m_-, m_+) \in \mathcal{M}_0^2} \mathcal{S}_{\text{bist}}(m_-, m_+).$$

For u in \mathcal{S} , let

$$I(u) = \bigcup_{x \in \mathbb{R}} \{(u(x), u'(x))\}$$

denote the trajectory (“image”) of this stationary solution in the phase space \mathbb{R}^{2n} of the Hamiltonian system (5).

For m in \mathcal{M}_0 , let $W^s(m, 0)$ denote the stable manifold of the equilibrium $(m, 0)$ for the Hamiltonian system (5), and let $W^u(m, 0)$ denote its unstable manifold.

It will be shown that, under certain hypotheses, bistable solutions approach the following subset of \mathbb{R}^{2n} , made of trajectories corresponding to stationary solutions of system (1) that are spatially homoclinic or heteroclinic to points of \mathcal{M}_0 :

$$\begin{aligned} I(\mathcal{S}_{\text{bist}}(\mathcal{M}_0)) &= \bigcup_{u \in \mathcal{S}_{\text{bist}}(\mathcal{M}_0)} I(u) \\ &= \left(\bigcup_{m \in \mathcal{M}_0} \{(m, 0)\} \right) \cup \left(\bigcup_{(m_-, m_+) \in \mathcal{M}_0^2} W^u(m_-, 0) \cap W^s(m_+, 0) \right). \end{aligned}$$

The shape of the set $I(\mathcal{S}_{\text{bist}}(\mathcal{M}_0))$ is illustrated on figure 2, for various familiar examples of potential V , in the scalar case $n = 1$.

2.7 Asymptotic energy of a bistable solution

The following preliminary result provides a definition of the asymptotic energy of a bistable solution.

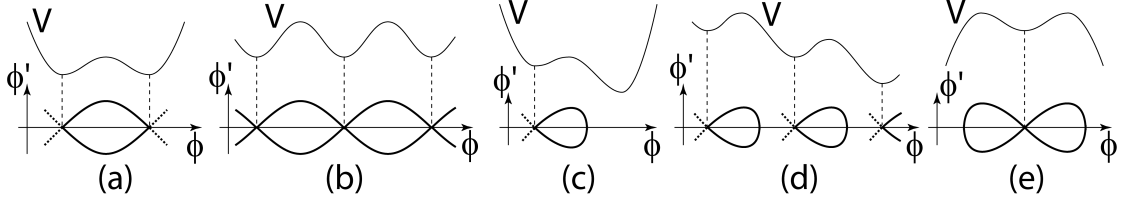


Figure 2: Shapes of familiar examples of potentials and of the corresponding phase portraits of system (5) governing stationary solutions of system (1): (a) the Allen-Cahn equation, (b) the over-damped sine-Gordon equation, (c) the Nagumo equation, (d) the over-damped sine-Gordon equation with constant forcing, and (e) the “subcritical” Allen-Cahn equation. The corresponding equations are briefly discussed in section 10 on page 53.

Proposition 1 (asymptotic energy). *Assume that V satisfies hypotheses (H_{coerc}) , (H_{norm}) , and (H_{min}) . Then, for every bistable solution $(x, t) \mapsto u(x, t)$ of system (1) connecting two minimum points of V in the level set $V^{-1}(\{0\})$, there exists a quantity \mathcal{E}_∞ in $\{-\infty\} \cup [0, +\infty)$ such that, for every sufficiently large positive quantity c ,*

$$\int_{-ct}^{ct} \left(\frac{1}{2} |u_x(x, t)|^2 + V(u(x, t)) \right) dx \rightarrow \mathcal{E}_\infty \quad \text{when } t \rightarrow +\infty.$$

Definition. If $(x, t) \mapsto u(x, t)$ is a bistable solution connecting two minimum points of V in the level set $V^{-1}(\{0\})$, let us call *asymptotic energy of this solution* the limit \mathcal{E}_∞ in $\{-\infty\} \cup [0, +\infty)$ given by this proposition.

Similarly, if a function u_0 in X is a bistable initial condition connecting two minimum points in the level set $V^{-1}(\{0\})$, let us call *asymptotic energy of u_0* the limit given by the proposition above for the solution corresponding to the initial condition u_0 , and let us denote by

$$\mathcal{E}_\infty[u_0]$$

this asymptotic energy.

2.8 Main result (first version): bistable solutions of finite asymptotic energy approach bistable stationary solutions

Let us recall the well-known definition of the distance between a point z_0 and a subset Σ of \mathbb{R}^{2n} :

$$\text{dist}(z_0, \Sigma) = \inf_{z \in \Sigma} |z - z_0|$$

where $|\cdot|$ denotes (say) the usual euclidean norm on \mathbb{R}^{2n} .

Theorem 1 (approach to the set of bistable stationary solutions). *Assume that V satisfies hypotheses (H_{coerc}) , (H_{norm}) , and (H_{min}) . Then, for every bistable solution*

$(x, t) \mapsto u(x, t)$ of system (1) connecting two minimum points of V in the level set $V^{-1}(\{0\})$, if the asymptotic energy of this solution is not $-\infty$, then both quantities

$$\sup_{x \in \mathbb{R}} |u_t(x, t)| \quad \text{and} \quad \sup_{x \in \mathbb{R}} \text{dist} \left(\left(u(x, t), u_x(x, t) \right), I(\mathcal{S}_{\text{bist}}(\mathcal{M}_0)) \right)$$

approach 0 when time approaches infinity.

This result will be reformulated below (Theorem 2 below) with a more accurate description of the asymptotic behaviour of the solution (and under an additional generic hypothesis on V).

If conversely the asymptotic energy of u_0 equals minus infinity, then the corresponding solution certainly takes values where the potential is negative when time increases, but no precise information on its behaviour will be given in this paper. In the companion paper [27] (following [26]), it is proved (only when the diffusion matrix \mathcal{D} is equal to identity) that in this case the solution displays travelling fronts invading the stable equilibria at both ends of space. Results of the same kind have been obtained (in a different setting limited to the scalar case $n = 1$) by Muratov and X. Zhong in [19].

2.9 Upper semi-continuity of the asymptotic energy

Notation. Let $X_{\text{bist}}(\mathcal{M}_0)$ denote the union of all initial conditions connecting minimum points of V in the level set $V^{-1}(\{0\})$; with symbols:

$$X_{\text{bist}}(\mathcal{M}_0) = \bigsqcup_{(m_-, m_+) \in \mathcal{M}_0^2} X_{\text{bist}}(m_-, m_+).$$

Definition. Proposition 1 above thus defines the *asymptotic energy functional*:

$$(7) \quad \begin{aligned} \mathcal{E}_\infty : X_{\text{bist}}(\mathcal{M}_0) &\rightarrow \{-\infty\} \sqcup \mathbb{R}_+ \\ u_0 &\mapsto \mathcal{E}_\infty[u_0] \end{aligned}$$

Exactly as for the (descendent) gradient flow of every regular function on a finite-dimensional manifold, the asymptotic energy is upper semi-continuous with respect to initial data, as stated by the following proposition. All its statement hold with respect to the topology induced on $X_{\text{bist}}(\mathcal{M}_0)$ by the X -norm and the topology induced on $\{-\infty\} \sqcup \mathbb{R}_+$ by the usual topology on $\{-\infty\} \sqcup \mathbb{R}$.

Proposition 2 (upper semi-continuity of asymptotic energy). *Assume that V satisfies hypotheses (H_{coerc}) , (H_{norm}) , and (H_{min}) . Then the asymptotic energy functional is upper semi-continuous; equivalently, for every real quantity E , the set*

$$\mathcal{E}_\infty^{-1}([E, +\infty)) = \{u_0 \in X_{\text{bist}}(\mathcal{M}_0) : \mathcal{E}_\infty[u_0] \geq E\}$$

is closed. In particular, the subset of $X_{\text{bist}}(\mathcal{M}_0)$ made of bistable initial conditions having a “non minus infinity” asymptotic energy is closed.

Let us mention here another result of the same nature: Theorem 2 of [26], stating that the speed of a travelling front invading a stable equilibrium is lower semi-continuous with respect to initial data.

2.10 Existence of homoclinic or heteroclinic stationary solutions and basin of attraction of a stable homogeneous stationary solution

A series of standard results can be recovered as direct consequences of Theorem 1 and Proposition 2. Those results deal with:

- existence of homoclinic or heteroclinic orbits of the Hamiltonian systems governing stationary solutions;
- the basin of attraction of the homogeneous stationary solution given by a minimum point of the potential (or the border of this basin of attraction).

To avoid disrupting the attention of the reader from the main results, these auxiliary results and their proofs are postponed until section 9 on page 49.

2.11 Normalization of bistable stationary solutions with respect to translation invariance and additional generic hypothesis

Due to space translation invariance, nonconstant stationary solutions of system (1) go by one-parameter families. For various reasons (in particular to state hypothesis (H_{disc}) below, that will be required for the next result) it is convenient to pick up a representative in each of these one-parameter families. This is done through the next definitions.

Let $\lambda_{V,\min}$ ($\lambda_{V,\max}$) denote the minimum (respectively, maximum) of all eigenvalues of the Hessian matrices of the potential V at minimum points of the level set $V^{-1}(\{0\})$. In other words, if $\sigma(D^2V(u))$ denotes the spectrum of the Hessian matrix of V at a point u in \mathbb{R}^n ,

$$\lambda_{V,\min} = \min_{m \in \mathcal{M}_0} \min(\sigma(D^2V(m))) \quad \text{and} \quad \lambda_{V,\max} = \max_{m \in \mathcal{M}_0} \max(\sigma(D^2V(m)))$$

(recall that the set \mathcal{M}_0 is finite). Obviously,

$$0 < \lambda_{V,\min} \leq \lambda_{V,\max} < +\infty.$$

Notation. For the remaining of this paper, let us fix a positive quantity d_{Esc} , sufficiently small so that, for every minimum point m of V in $V^{-1}(\{0\})$ and for all u in \mathbb{R}^n satisfying $|u - m|_{\mathcal{D}} \leq d_{\text{Esc}}$, every eigenvalue λ of $D^2V(u)$ satisfies:

$$(8) \quad \frac{\lambda_{V,\min}}{2} \leq \lambda \leq 2\lambda_{V,\max}.$$

The reason for the subscript “Esc” in this notation is that this distance d_{Esc} will be used to “track” the position in space where a solution “escapes” a neighbourhood of a minimum point of V (this position is called “leading edge” by Cyrill B. Muratov [17–19]). Inside this neighbourhood, the potential essentially behaves like a positive definite quadratic form; and every nonconstant stationary solution, connecting two minimum points in the level set $V^{-1}(\{0\})$, “escapes” at least at distance d_{Esc} (for the $|\cdot|_{\mathcal{D}}$ -norm) from each of these two points (even if these two points are equal) at some position of

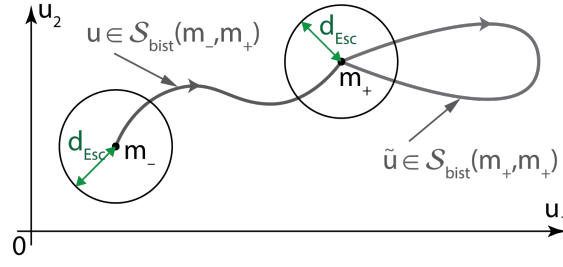


Figure 3: Nonconstant stationary solutions in $\mathcal{S}_{\text{bist}}(\mathcal{M}_0)$ escape at least at a $|\cdot|_{\mathcal{D}}$ -distance d_{Esc} of their limits at $\pm\infty$.

space (see figure 3 and Lemma 19 on page 61). In other words, for every pair (m_-, m_+) of points of \mathcal{M}_0 and for every nonconstant stationary solution u connecting m_- to m_+ ,

$$\sup_{x \in \mathbb{R}} |u(x) - m_-|_{\mathcal{D}} > d_{\text{Esc}} \quad \text{and} \quad \sup_{x \in \mathbb{R}} |u(x) - m_+|_{\mathcal{D}} > d_{\text{Esc}}.$$

Let us mention that there is nothing profound behind the choice of using the $|\cdot|_{\mathcal{D}}$ rather than the usual Euclidian norm of \mathbb{R}^n to define this escape distance. The sole reason is that Lemma 19 on page 61 is more natural with this definition.

For every nonconstant stationary solution connecting two minimum points of V in the level set $V^{-1}(\{0\})$, a unique translate of this solution can be picked up by demanding that, say, the translate be exactly at distance d_{Esc} of his left-end limit m_- at $x = 0$, and closer for every negative x (see figure 4). Here is a more formal definition. For (m_-, m_+)

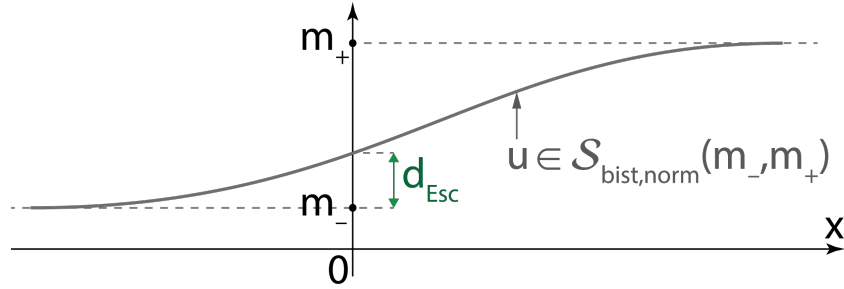


Figure 4: Normalized stationary solution.

in \mathcal{M}_0^2 , let us consider the set of *normalized bistable stationary solutions connecting m_- to m_+* :

$$\begin{aligned} \mathcal{S}_{\text{bist, norm}}(m_-, m_+) &= \{u \in \mathcal{S}_{\text{bist}}(m_-, m_+) : \\ &\quad |u(0) - m_-|_{\mathcal{D}} = d_{\text{Esc}} \quad \text{and} \quad |u(x) - m_-|_{\mathcal{D}} < d_{\text{Esc}} \quad \text{for all } x < 0\} \end{aligned}$$

and let

$$\mathcal{S}_{\text{bist, norm}}(\mathcal{M}_0) = \bigcup_{(m_-, m_+) \in \mathcal{M}_0^2} \mathcal{S}_{\text{bist, norm}}(m_-, m_+).$$

13

- a bistable stationary solution u_1 connecting m_- to m_+
- and a \mathcal{C}^1 -function $t \mapsto x_1(t)$ defined on \mathbb{R}_+ and satisfying $x_1'(t) \rightarrow 0$ when t approaches $+\infty$

such that, for every real quantity x and every nonnegative time t ,

$$\mathcal{T}(x, t) = u_1(x - x_1(t));$$

3. if q is not smaller than 2, then there exists $q - 1$ minimum points m_1, \dots, m_{q-1} of V (not necessarily distinct), all in the level set $V^{-1}(\{h\})$, and if we denote m_- by m_0 and m_+ by m_q , then for each integer i in $\{1, \dots, q\}$, there exists:

- a bistable stationary solution u_i connecting m_{i-1} to m_i
- and a \mathcal{C}^1 -function $t \mapsto x_i(t)$ defined on \mathbb{R}_+ and satisfying $x_i'(t) \rightarrow 0$ when t approaches $+\infty$

such that, for every integer i in $\{1, \dots, q - 1\}$,

$$x_{i+1}(t) - x_i(t) \rightarrow +\infty \quad \text{when} \quad t \rightarrow +\infty,$$

and such that, for every real quantity x and every nonnegative time t ,

$$\mathcal{T}(x, t) = m_0 + \sum_{i=1}^q \left[u_i(x - x_i(t)) - m_{i-1} \right].$$

Obviously, item 2 may have been omitted in this definition, since it fits with item 3 with q equals 1. For sake of generality this definition was given for any level set of the potential, however in the present paper it will only be used for the level set $V^{-1}(\{0\})$.

Definition (energy of a bistable stationary solution). Let $x \mapsto u(x)$ be a bistable stationary solution connecting two local minima m_- and m_+ of V , and let h denote the quantity $V(m_+)$ (which is equal to $V(m_-)$). The quantity

$$\mathcal{E}[u] = \int_{\mathbb{R}} \left(\frac{|u'(x)|^2}{2} + V(u(x)) - h \right) dx$$

is called the *energy of the (bistable) stationary solution u* . Observe that this integral converges, since $u(x)$ approaches its limits m_- and m_+ at both ends of space at an exponential rate.

This definition will be used in this paper only in the case where h equals 0.

Definition (energy of a standing terrace). Let h denote a real quantity and let \mathcal{T} denote a standing terrace of bistable stationary solutions connecting two local minima of V in the level set $V^{-1}(\{h\})$. With the notation of the two definitions above, the quantity $\mathcal{E}[\mathcal{T}]$ defined by:

1. if q equals 0, then $\mathcal{E}[\mathcal{T}] = 0$,

2. if q equals 1, then $\mathcal{E}[\mathcal{T}] = \mathcal{E}[u_1]$,
3. if q is not smaller than 2, then $\mathcal{E}[\mathcal{T}] = \sum_{i=1}^q \mathcal{E}[u_i]$,

is called the *energy of the standing terrace* \mathcal{T} .

Again, this definition will be used in this paper only in the case where h equals 0.

The terminology “propagating terrace” was introduced by A. Ducrot, T. Giletti, and H. Matano in [6] (and subsequently used by P. Poláčik, [20–22]) to denote a stacked family (a layer) of travelling fronts in a (scalar) reaction-diffusion equation. This led the author to introduce the analogous “standing terrace” terminology above, because this terminology is convenient to denote an object otherwise requiring a quite long description, and because it provides a convenient homogeneity in the formulation of the results of [27] describing the asymptotic behaviour of all bistable solutions of systems like (1), since this behaviour involves altogether two “propagating terraces” (one to the left and one to the right) and a “standing terrace” in between. This terminology is also used in the companion papers [28, 29].

The author hopes that these advantages balance some drawbacks of this terminological choice. Like the fact that the word “terrace” is probably more relevant in the scalar case $n = 1$ (see the pictures in [6, 21]) than in the more general case of systems considered here. Or the fact that the definitions above and in [27] are different from the original definition of [6] in that they involve not only the profiles of particular (standing or travelling) solutions, but also their positions (denoted above by $x_i(t)$).

To finish, observe that in the present context terraces are only made of bistable solutions, by contrast with the propagating terraces introduced and used by the authors cited above; that standing terraces are approached by solutions but are (in general) not solutions themselves; and that a standing terrace may be nothing but a single stable homogeneous equilibrium (when q equals 0).

2.13 Main result, second version: bistable solutions of finite asymptotic energy converge towards a standing terrace of bistable stationary solutions

Compared to Theorem 1 above, the following theorem (the second main result of this paper) provides a more precise description of the asymptotic behaviour of the solution under consideration, by taking advantage of the additional (generic) hypothesis (H_{disc}) .

Theorem 2 (convergence towards a standing terrace of bistable stationary solutions). *Assume that the potential V satisfies hypotheses (H_{coerc}) , (H_{norm}) , (H_{min}) , and (H_{disc}) . Then, every bistable solution of system (1) connecting local minimum points in the level set $V^{-1}(\{0\})$ and having a finite asymptotic energy approaches (uniformly in space, when time approaches $+\infty$) a standing terrace of bistable stationary solutions. In addition, the asymptotic energy of the solution equals the energy of the standing terrace.*

With symbols, this theorem can be reformulated as follows. Let m_- and m_+ be two minimum points of V in the level set $V^{-1}(\{0\})$, and let u_0 be a bistable initial condition

connecting these two points. Assume that

$$\mathcal{E}_\infty[u_0] > -\infty.$$

Then there exists a standing terrace \mathcal{T} of bistable stationary solutions, connecting m_- to m_+ , such that

$$\sup_{x \in \mathbb{R}} |u(x, t) - \mathcal{T}(x, t)| \rightarrow 0 \quad \text{when } t \rightarrow +\infty.$$

In addition,

$$\mathcal{E}_\infty[u_0] = \mathcal{E}[\mathcal{T}]$$

(and this quantity is nonnegative).

Obviously in this theorem the profiles involved in the standing terrace \mathcal{T} (and their order) are uniquely defined by the solution (uniquely if profiles are taken in $\mathcal{S}_{\text{bist, norm}}(\mathcal{M}_0)$ and uniquely up to space translation if they are taken in $\mathcal{S}_{\text{bist}}(\mathcal{M}_0)$), but not their positions $t \mapsto x_i(t)$.

2.14 Additional remarks and comments

2.14.1 Examples

Elementary examples corresponding to the potentials illustrated on figure 2 (in the scalar case $n = 1$) are discussed in section 10 on page 53.

2.14.2 Convergence for a stronger topology

Due to the smoothing properties of system (1) (see subsection 3.2), convergence towards the standing terrace in Theorem 2 holds with respect to the $\mathcal{C}_b^k(\mathbb{R}, \mathbb{R}^n)$ -norm, where k is the largest integer such that $V(\cdot)$ is of classe \mathcal{C}^k .

2.14.3 Long range interaction between bistable stationary solutions

It is possible, under some additional (transversality) hypotheses, to study more precisely the long-range interaction between the bistable stationary solutions involved in the standing terrace (provided by Theorem 2) describing the asymptotic behaviour of the solution (in the case $q \geq 2$), and to obtain explicit expressions for the asymptotics (at first order) of the positions $\xi_i(t)$, $i \in \{1, \dots, q\}$, when $t \rightarrow +\infty$? (see S. I. Ei's paper [7] and conjecture p. 59 of Béthuel, Orlandi, and Smets [4]).

Since these stationary solutions must go (slowly) away from one another, the first order interaction term between two successive stationary solutions u_i and u_{i+1} , $i \in \{1, \dots, q-1\}$ should be repulsive, and this should give some restrictions on the families (u_1, \dots, u_q) that can actually be involved in the standing terrace approached by the solution. Elementary examples are discussed in section 10 on page 53, but general statements and rigorous proofs, taking into account the fact that those individual stationary solutions are not necessarily stable, are beyond the scope of this paper.

2.14.4 Quantitative estimates on the rate of convergence

As already mentioned in the introduction, there is a huge amount of literature about relaxation of “bistable” solutions for systems like (1). In particular, this phenomenon is investigated by Béthuel, Orlandi, and Smets in [4]. They obtain quantitative (thus more precise) estimates on the rate of convergence of these solutions towards the set of bistable solutions. Although the hypotheses made by these authors are slightly more restrictive (they consider a potential taking only nonnegative values, solutions with finite energy, and a diffusion matrix equal to identity), it is likely that their approach applies to the hypotheses considered here, and provides alternative proofs (and extensions to more precise quantitative statements) of the results stated above. The approach developed in the present paper is by contrast purely qualitative (no information is given about the rate of convergence).

2.14.5 Extension to the damped hyperbolic case

It is likely that similar results hold for the damped hyperbolic system

$$(9) \quad \alpha u_{tt} + u_t = -\nabla V(u) + u_{xx},$$

obtained by adding an inertial term αu_{tt} (where α is a positive non necessarily small quantity) to the parabolic system (1) considered here. Some work in this direction was done in the author’s previous paper [25], where both parabolic and hyperbolic cases were treated simultaneously (only in the scalar case $n = 1$). The much more difficult problem of global convergence towards travelling fronts was solved by Gallay and Joly in [12], still in the scalar case $n = 1$. These results have recently been extended to hyperbolic systems, [28].

2.14.6 Unsolved questions

Besides the questions asked in section 13 and sub-subsections 2.14.3 and 2.14.4 above, here are some additional (and, to my knowledge, open) questions that raise naturally from the statements above.

1. Do Theorem 1 and Theorem 2 still hold without hypothesis (H_{\min}) (stating that all critical points in the level set $V^{-1}(\{0\})$ of the potential are nondegenerate local minima) ? (this question is twofold: hypothesis (H_{\min}) may be relaxed assuming that those critical points are still local minimum points but possibly degenerate ones, or dropping any additional hypothesis about these critical points).
2. Does Theorem 2 still hold without hypothesis (H_{disc}) (stating that the set of normalized bistable stationary solutions of zero Hamiltonian is totally disconnected in X) ? For instance, does it hold for the $O(2)$ -symmetric “real Ginzburg–Landau” potentials (see figure 6):

$$V : \mathbb{C} \simeq \mathbb{R}^2 \rightarrow \mathbb{R}, \quad z \mapsto \frac{|z|^2}{2} - \frac{|z|^4}{4} \quad \text{or} \quad z \mapsto \frac{|z|^2}{2} - \frac{4}{\sqrt{3}} \frac{|z|^4}{4} + \frac{|z|^6}{6} \quad ?$$

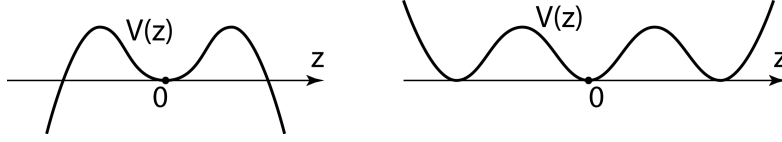


Figure 6: Graphs of the restrictions to the real line of the two examples of potentials $z \mapsto V(z)$ for which hypothesis (H_{disc}) does not hold.

3. Is it possible to construct an example where Theorem 2 holds, where the number q of items involved in the standing terrace equals 1, but where the “position” $\xi(t)$ does not converge when time approaches $+\infty$? (note that this surely requires that the stationary solution be “degenerated” in the sense that it be not a hyperbolic equilibrium for the semi-flow of system (1)). On the other hand, does $\xi(t)$ always converge when V is analytic? (see [30]).

2.15 Organization of the paper

- The next section 3 is devoted to some preliminaries (functional framework, existence of solutions, preliminary computations on spatially localized functionals, notation).
- Preliminary results on spatial asymptotics of bistable solutions are stated and proved in section 4 on page 23.
- Proof of Theorem 1 really starts in section 5 on page 34: the results obtained in section 4 ensure that a (properly localized) energy functional is almost decreasing with time (the “flux” term approaches 0 at an exponential rate). The asymptotic energy of a solution can therefore be defined as the limit of this functional when time approaches infinity (section 5).
- The approach to the set $I(\mathcal{S}_{\text{bist}}(\mathcal{M}_0))$ is argued in section 6 on page 37. The proof goes through several steps. First the assumption that the asymptotic energy of the solution is not equal to $-\infty$ will be used to prove that the time derivative $u_t(x, t)$ of the solution approaches 0, uniformly in space, when time approaches infinity (subsection 6.1). The next step is to prove that the “Hamiltonian energy” (6) of the solution goes to 0, uniformly with respect to x , at least for a (growing, unbounded) sequence $(t_k)_k$ of values of time. Another step is to prove that this convergence occurs for all times t approaching $+\infty$ (it follows at this stage that the asymptotic energy is nonnegative if not minus infinity, and this finishes to prove Proposition 1). Then the approach to the set $I(\mathcal{S}_{\text{bist}}(\mathcal{M}_0))$ is completed (and this completes the proof of Theorem 1).
- The more precise conclusions of Theorem 2 are proved in section 7 on page 43.
- Proposition 2 is proved in section 8 on page 47.

The remaining sections can be viewed as appendices.

- Section 9 on page 49 is devoted to the statements and proofs of standard results (Corollaries 3 to 6) concerning existence of homoclinic or heteroclinic stationary solutions and the basin of attraction of a stable homogeneous solution, as direct consequences of Theorem 1 and Proposition 2.
- Elementary examples illustrating the results — and the questions raised — are discussed in section 10 on page 53.
- The proof of the existence of an attracting ball for the semi-flow follows from the coercivity hypothesis (H_{coerc}) and is given in section 11 on page 56.
- Section 12 on page 60 is devoted to two lemmas concerning stationary solutions of system (1), extensively used to prove the approach to the set $I(\mathcal{S}_{\text{bist}}(\mathcal{M}_0))$ in section 6.
- Finally, a rough discussion of the map between initial conditions and the space of asymptotic patterns (and the regularity of this map) is carried out in section 13 on page 64.

3 Preliminaries

3.1 Functional framework

For u in $H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^n)$, let

$$\|u\|_{H_{\text{ul}}^1(\mathbb{R}, \mathbb{R}^n)} = \sup_{\xi \in \mathbb{R}} \left(\int_{\xi}^{\xi+1} (|u(x)|^2 + |u'(x)|^2) dx \right)^{1/2} = \sup_{\xi \in \mathbb{R}} \|u\|_{H^1([\xi, \xi+1], \mathbb{R}^n)} \leq \infty,$$

and let us consider the uniformly local Sobolev space X defined as

$$\begin{aligned} X &= H_{\text{ul}}^1(\mathbb{R}, \mathbb{R}^n) \\ &= \{u \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^n) : \|u\|_{H_{\text{ul}}^1(\mathbb{R}, \mathbb{R}^n)} < \infty \text{ and } \lim_{\xi \rightarrow 0} \|T_{\xi}u - u\|_{H_{\text{ul}}^1(\mathbb{R}, \mathbb{R}^n)} = 0\}. \end{aligned}$$

As already mentioned in subsection 2.2, this space is the most convenient with respect to the estimates on localized energy and L^2 -norm that are used all along the paper. However, due to the smoothing properties of system (1), the choice of the functional framework is not crucial, and every statement of this paper remains true if X denotes, instead of $H_{\text{ul}}^1(\mathbb{R}, \mathbb{R}^n)$, the more familiar Banach space $\mathcal{C}_{\text{b}}^1(\mathbb{R}, \mathbb{R}^n)$ of functions of class \mathcal{C}^1 that are uniformly bounded together with their first derivative.

3.2 Global existence of solutions and attracting ball for the semi-flow

Since V is assumed to be of class at least \mathcal{C}^2 , the map $v \mapsto \nabla V(v)$ is of class at least \mathcal{C}^1 , and therefore the nonlinearity $u(\cdot) \mapsto -\nabla V(u(\cdot))$ in system (1) is locally Lipschitz in X . Thus local existence of solutions in that space follows from general results (see for instance Henry's book [16]).

More precisely, for every u_0 in X , system (1) has a unique (mild) solution $t \mapsto S_t u_0$ in $\mathcal{C}^0([0, T_{\max}), X)$ with initial data u_0 . This solution depends continuously on the initial condition u_0 and is defined up to a (unique) maximal time of existence $T_{\max} = T_{\max}[u_0]$ in $(0, +\infty]$.

For every integer k the space $\mathcal{C}_b^k(\mathbb{R}, \mathbb{R}^n)$ is equipped with the usual norm:

$$\|v\|_{\mathcal{C}_b^k(\mathbb{R}, \mathbb{R}^n)} = \sup_{x \in \mathbb{R}} |v(x)| + \sup_{x \in \mathbb{R}} |v'(x)| + \cdots + \sup_{x \in \mathbb{R}} |v^{(k)}(x)|.$$

The following global existence result (proved in section 11) follows from the coercivity hypothesis (H_{coerc}) on the potential V .

Lemma 1 (global existence of solutions and attracting ball). *For every function u_0 in X , the solution $t \mapsto S_t u_0$ of system (1) with initial data u_0 is defined up to $+\infty$ in time. In addition, there exists*

- a positive quantity R_{att} (“radius of attracting ball for the $L^\infty(\mathbb{R}, \mathbb{R}^n)$ -norm”), depending only on V and \mathcal{D} ,

and, for every positive quantity R (“initial radius for the X -norm”) there exist

- a positive quantity $R_{\max}(R)$ (“radius of maximal excursion for the $L^\infty(\mathbb{R}, \mathbb{R}^n)$ -norm”), depending only on V and \mathcal{D} and R ,
- and a positive quantity $T_{\text{att}}(R)$ (“delay to enter attracting ball”), depending only on V and \mathcal{D} and R ,

such that, if

$$\|u_0\|_X \leq R,$$

then

$$\sup_{t \geq 0} \sup_{x \in \mathbb{R}} |(S_t u_0)(x)| \leq R_{\max}(R) \quad \text{and} \quad \sup_{t \geq T_{\text{att}}(R)} \sup_{x \in \mathbb{R}} |(S_t u_0)(x)| \leq R_{\text{att}}.$$

Thus, the ball of radius R_{att} and center at the origin of \mathbb{R}^n is an attractive ball for the $L^\infty(\mathbb{R}, \mathbb{R}^n)$ -norm for the semi-flow, and the time required to enter this attracting ball is uniform in the sense that it depends only on the size (and not on other features) of the initial condition.

In addition, system (1) has smoothing properties (Henry [16]). Due to these properties, since V is of class \mathcal{C}^k , every solution $t \mapsto S_t u_0$ in $\mathcal{C}^0([0, +\infty), X)$ actually belongs to

$$\mathcal{C}^0((0, +\infty), \mathcal{C}_b^{k+1}(\mathbb{R}, \mathbb{R}^n)) \cap \mathcal{C}^1((0, +\infty), \mathcal{C}_b^{k-1}(\mathbb{R}, \mathbb{R}^n)),$$

and, for every positive quantity ε , the following quantities

$$(10) \quad \sup_{t \geq \varepsilon} \|S_t u_0\|_{\mathcal{C}_b^{k+1}(\mathbb{R}, \mathbb{R}^n)} \quad \text{and} \quad \sup_{t \geq \varepsilon} \left\| \frac{d(S_t u_0)}{dt}(t) \right\|_{\mathcal{C}_b^{k-1}(\mathbb{R}, \mathbb{R}^n)}$$

are finite.

3.3 Time derivative of (localized) energy and L^2 -norm of a solution

Let $(x, t) \mapsto u(x, t)$ denote a solution of system (1). Key ingredients in the proofs rely on appropriate combinations of the two most natural functionals to consider, namely the energy (Lagrangian) and the L^2 -norm of the solution:

$$\int_{\mathbb{R}} \left(\frac{|u_x(x, t)|_{\mathcal{D}}^2}{2} + V(u(x, t)) \right) dx \quad \text{and} \quad \int_{\mathbb{R}} \frac{u(x, t)^2}{2} dx.$$

Of course, since the a priori bounds stated in subsection 3.2 above just ensure that the two integrands are bounded, it is necessary to localize these integrands to ensure the convergence of the integrals. Let $x \mapsto \psi(x)$ denote a function in the space $W^{2,1}(\mathbb{R}, \mathbb{R})$ (that is a function belonging to $L^1(\mathbb{R})$, together with its first and second derivatives). Then, the time derivatives of the two aforementioned functionals — localized by $\psi(x)$ — read:

$$(11) \quad \frac{d}{dt} \int_{\mathbb{R}} \psi \left(\frac{|u_x|_{\mathcal{D}}^2}{2} + V(u) \right) dx = \int_{\mathbb{R}} (-\psi u_t^2 - \psi' \mathcal{D}u_x \cdot u_t) dx$$

and

$$(12) \quad \begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \psi \frac{u^2}{2} dx &= \int_{\mathbb{R}} \left(\psi (-u \cdot \nabla V(u) - |u_x|_{\mathcal{D}}^2) - \psi' u \cdot \mathcal{D}u_x \right) dx \\ &= \int_{\mathbb{R}} \left(\psi (-u \cdot \nabla V(u) - |u_x|_{\mathcal{D}}^2) + \psi'' \frac{|u|_{\mathcal{D}}^2}{2} \right) dx. \end{aligned}$$

Here are some basic observations about these expressions.

- The variation of the (localized) energy is the sum of a (nonpositive) “dissipation” term and a additional “flux” term.
- The variation of the (localized) L^2 -norm is similarly made of two “main” terms and an additional “flux” term. Among the two main terms, the second one is nonpositive, and so is the first one if the quantity $u \cdot \nabla V(u)$ is positive, that is:
 - for $|u|$ large (according to the coercivity hypothesis (H_{coerc}) on V);
 - for $|u|$ small if the origin $0_{\mathbb{R}^n}$ of \mathbb{R}^n is a minimum point of V , say if $0_{\mathbb{R}^n}$ is in the set \mathcal{M}_0 .
- The second integration by parts that is performed on the last term of the expression (12) of the time derivative of the L^2 -functional will lead to slightly simpler calculations, but is not essential.
- The slower the weight function ψ varies, the smaller the flux terms are. More precisely, it seems relevant to choose ψ as a function satisfying, for a small positive quantity ε ,

$$|\psi'(x)| \leq \varepsilon \psi(x) \quad \text{and} \quad |\psi''(x)| \leq \varepsilon \psi(x) \quad \text{for all } x \text{ in } \mathbb{R}.$$

This way, if ε is small enough, the flux terms might very well be “dominated” by the other terms of the right-hand sides of equalities (11) and (12).

- An appropriate combination of these two functionals might display coercivity properties, again for $|u|$ large (according to the coercivity hypothesis (H_{coerc}) on V) and for $|u|$ small if $0_{\mathbb{R}^n}$ is in the set \mathcal{M}_0 .

These observations will be put in practice several times along the following pages:

1. to prove the existence of an attracting ball for the flow (section 11);
2. to gain some control on the spatial asymptotics of bistable solutions (sections 4 and 8);
3. to state the approximate decrease of localized energies (sections 5 and 8). For those localized energies the weight function that will be used (denoted by χ instead of ψ) will depend not only on x but also on t , thus the right-hand side of equality (11) will comprise an additional “flux” term with weight χ_t .

3.4 Miscellanea

3.4.1 Notation for the eigenvalues of the diffusion matrix

Let $\lambda_{\mathcal{D},\min}$ ($\lambda_{\mathcal{D},\max}$) denote the smallest (respectively, largest) of the eigenvalues of the diffusion matrix \mathcal{D} ; obviously,

$$0 < \lambda_{\mathcal{D},\min} \leq \lambda_{\mathcal{D},\max}.$$

3.4.2 Estimates derived from the definition of the “escape distance”

For every minimum point m in \mathcal{M}_0 and every vector v in \mathbb{R}^n satisfying $|v - m|_{\mathcal{D}} \leq d_{\text{Esc}}$, it follows from inequalities (8) on page 11 that

$$(13) \quad \begin{aligned} \frac{\lambda_{V,\min}}{4}(u - m)^2 &\leq V(u) && \leq \lambda_{V,\max}(u - m)^2, \\ \frac{\lambda_{V,\min}}{2}(u - m)^2 &\leq (u - m) \cdot \nabla V(u) && \leq 2\lambda_{V,\max}(u - m)^2. \end{aligned}$$

3.4.3 Minimum of the convexities of the lower quadratic hulls of the potential at local minimum points

For the computations carried in the next section 4, it will be convenient to introduce the quantity $q_{\text{low-hull}}$ defined as the minimum of the convexities of the lower quadratic hulls of V at the points of \mathcal{M}_0 (see figure 7). With symbols:

$$q_{\text{low-hull}} = \min_{m \in \mathcal{M}_0} \min_{u \in \mathbb{R}^n \setminus \{m\}} \frac{V(u)}{(u - m)^2}.$$

This definition ensures (as obviously displayed by figure 7) that, for every m in the set \mathcal{M}_0 and for all u in \mathbb{R}^n ,

$$(14) \quad V(u) - q_{\text{low-hull}}(u - m)^2 \geq 0.$$

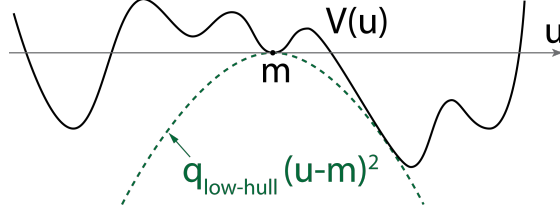


Figure 7: Lower quadratic hull of the potential at a minimum point (definition of the quantity $q_{\text{low-hull}}$).

Let us consider the following quantity (it will be used as the *weighting of the energy* in the firewall function defined in subsection 4.2):

$$w_{\text{en}} = \frac{1}{\max(1, -4 q_{\text{low-hull}})}.$$

It follows from this definition that, for every m in the set \mathcal{M}_0 and for all u in \mathbb{R}^n ,

$$(15) \quad w_{\text{en}} V(u) + \frac{(u - m)^2}{4} \geq 0.$$

4 Spatial asymptotics of bistable solutions

The aim of this section is to prove the following proposition. It states that if an initial condition is sufficiently close to minimum points of V at both ends of space, then the corresponding solution is “bistable” in the sense of the definition of subsection 2.3, and the rate at which it converges (as time increases) to the minimum points at both ends of space is exponential. The proposition introduces a quantity $c_{\text{no-inv}}$ (“no-invasion speed”) sufficiently large so that the domains at both ends of space where the solution is close to the minimum points cannot be “invaded” at a speed as large as this quantity. Actually, not only this proposition but also intermediate definitions and results that will be stated below along the proof will be used in the next sections.

For the “hyperbolic” version of this proposition, see Proposition 3.2 p. 114 of [12].

Proposition 3 (sufficient condition for bistability). *There exist positive quantities r and $c_{\text{no-inv}}$ and ν , depending only on V and \mathcal{D} such that the following assertion holds. If (m_-, m_+) is a pair of minimum points of V in the level set $V^{-1}(\{0\})$ and u_0 is a function in X satisfying:*

$$(16) \quad \limsup_{x \rightarrow -\infty} \int_{x-1}^x \left((u_0(y) - m_-)^2 + u_0'(y)^2 \right) dy \leq r^2$$

and

$$\limsup_{x \rightarrow +\infty} \int_x^{x+1} \left((u_0(y) - m_+)^2 + u_0'(y)^2 \right) dy \leq r^2,$$

then there exists a positive quantity $K(u_0)$ (depending on V , \mathcal{D} , and the initial data u_0) such that the solution $(x, t) \mapsto u(x, t)$ of system (1) with initial data u_0 satisfies the

following estimates:

$$\sup_{x \in (-\infty, -c_{\text{no-inv}} t]} |u(x, t) - m_-| \leq K(u_0) \exp(-\nu t),$$

and

$$\sup_{x \in [c_{\text{no-inv}} t, +\infty)} |u(x, t) - m_+| \leq K(u_0) \exp(-\nu t).$$

In particular, this solution is a bistable solution connecting m_- to m_+ .

The following corollary follows readily from this proposition.

Corollary 1 (bistable is open). *For every pair (m_-, m_+) of minimum points of V in the level set $V^{-1}(\{0\})$, the set of bistable initial conditions connecting m_- to m_+ is nonempty and open in X .*

The next subsections display strong similarities with section 3 of previous paper [26], although the presentation and hypotheses are slightly different.

4.1 Setup

Without loss of generality, it is sufficient to prove Proposition 3 only at one end of space (say on the right end of space) and assuming (to simplify the notation) that the minimum point m_+ is the origin $0_{\mathbb{R}^n}$ of \mathbb{R}^n .

Thus let us assume that $0_{\mathbb{R}^n}$ is a minimum point of V in the level set $V^{-1}(\{0\})$. Let R be a positive quantity (upper bound on the initial data for the X -norm) and u_0 be a function in X satisfying

$$\|u_0\|_X \leq R,$$

and let $(x, t) \mapsto u(x, t)$ denote the solution of system (1) with initial data u_0 . According to Lemma 1, there exists a quantity $R_{\max, \infty}(R)$ (maximal radius of excursion for the L^∞ -norm), depending only on V and \mathcal{D} and R , such that

$$(17) \quad \sup_{t \in [0, +\infty)} \sup_{x \in \mathbb{R}} |u(x, t)| \leq R_{\max, \infty}(R).$$

No more assumptions are made on the solution at this stage. The next subsections 4.2 to 4.5 are devoted to intermediate steps, and the proof of Proposition 3 will follow in subsection 4.6; only then will the assumptions (16) be made.

4.2 Firewal function and its time derivative

The proof relies on the definition of a functional that is an appropriate combination of the energy and the L^2 -norm of the solution, localized by an appropriate weight function (see subsection 3.3 and comments therein). As already mentioned in subsection 3.3, the key points are:

- to choose the relative weightings for energy and L^2 -norm in such a way that the resulting function has coercivity properties;

- to choose a weight function that varies sufficiently slowly in order to recover from expressions (11) and (12) some decrease of the resulting function.

Concerning the first of these two points, the quantity w_{en} defined in sub-subsection (3.4.3) is a convenient weighting for energy, as can be seen from inequality (15) satisfied by this quantity. Concerning the second point, let κ denote a positive quantity, sufficiently small so that

$$(18) \quad \frac{w_{\text{en}} \kappa^2 \lambda_{\mathcal{D},\max}}{4} \leq \frac{1}{2} \quad \text{and} \quad \frac{\kappa^2 \lambda_{\mathcal{D},\max}}{2} \leq \frac{\lambda_{V,\min}}{4}$$

(those properties will be used to prove inequality (21) below), namely:

$$\kappa = \min \left(\sqrt{\frac{2}{w_{\text{en}} \lambda_{\mathcal{D},\max}}}, \sqrt{\frac{\lambda_{V,\min}}{2 \lambda_{\mathcal{D},\max}}} \right),$$

and let us consider the weight function ψ defined by:

$$\psi(x) = \exp(-\kappa|x|).$$

For ξ in \mathbb{R} , let $T_\xi \psi$ denote the translate of ψ by ξ , that is the function defined by:

$$T_\xi \psi(x) = \psi(x - \xi)$$

(see figure 8). For all ξ in \mathbb{R} and t in $[0, +\infty)$, let

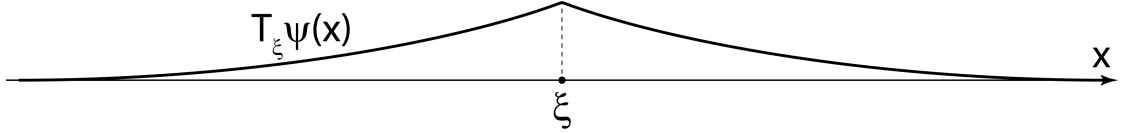


Figure 8: Graph of the weight function $x \mapsto T_\xi \psi(x)$ used to define the firewall function $\mathcal{F}(\xi, t)$. The slope is small, according to the definition of κ .

$$\mathcal{F}(\xi, t) = \int_{\mathbb{R}} T_\xi \psi(x) \left(w_{\text{en}} \left(\frac{|u_x(x, t)|_{\mathcal{D}}^2}{2} + V(u(x, t)) \right) + \frac{u(x, t)^2}{2} \right) dx.$$

According to inequality (15) satisfied by w_{en} , this quantity is coercive in the following sense: for all ξ in \mathbb{R} and all t in $[0, +\infty)$,

$$(19) \quad \mathcal{F}(\xi, t) \geq \min \left(\frac{w_{\text{en}}}{2}, \frac{1}{4} \right) \int_{\mathbb{R}} T_\xi \psi(x) (|u_x(x, t)|_{\mathcal{D}}^2 + u(x, t)^2) dx.$$

This quantity will play the role of a “firewall”, in the sense that its approximate decrease will enable to control the solution in the part of space where it is not too far from the minimum $0_{\mathbb{R}^n}$ (and consequently to control the flux term in the derivative of the localized energy in the next section). The notation \mathcal{F} relates to this interpretation. This approximate decrease is formalized by the following lemma.

For t in $[0, +\infty)$, let us consider the set (the domain of space where the solution “Escapes” at a certain distance from $0_{\mathbb{R}^n}$):

$$\Sigma_{\text{Esc}}(t) = \{x \in \mathbb{R} : |u(x, t)|_{\mathcal{D}} > d_{\text{Esc}}\}.$$

Lemma 2 (firewall decrease up to pollution term). *There exist positive quantities $\nu_{\mathcal{F}}$ (depending only on V and \mathcal{D}) and $K_{\mathcal{F}}(R)$ (depending only on V and \mathcal{D} and R) such that, for all ξ in \mathbb{R} and all t in $[0, +\infty)$,*

$$(20) \quad \partial_t \mathcal{F}(\xi, t) \leq -\nu_{\mathcal{F}} \mathcal{F}(\xi, t) + K_{\mathcal{F}}(R) \int_{\Sigma_{\text{Esc}}(t)} T_{\xi} \psi(x) dx.$$

Proof. It follows from expressions (11) and (12) that, for all ξ in \mathbb{R} and all t in $[0, +\infty)$,

$$\begin{aligned} \partial_t \mathcal{F}(\xi, t) &= \int_{\mathbb{R}} T_{\xi} \psi \left(-w_{\text{en}} u_t^2 - u \cdot \nabla V(u) - |u_x|_{\mathcal{D}}^2 \right) dx - \int_{\mathbb{R}} T_{\xi} \psi' (w_{\text{en}} \mathcal{D} u_x \cdot u_t) dx \\ &\quad + \int_{\mathbb{R}} T_{\xi} \psi'' \frac{|u|_{\mathcal{D}}^2}{2} dx. \end{aligned}$$

Since

$$|\psi'(\cdot)| \leq \kappa \psi(\cdot) \quad \text{and} \quad \psi''(\cdot) \leq \kappa^2 \psi(\cdot)$$

(indeed $\psi''(\cdot)$ equals $\kappa^2 \psi(\cdot)$ plus a Dirac mass of negative weight), it follows that

$$\partial_t \mathcal{F}(\xi, t) \leq \int_{\mathbb{R}} T_{\xi} \psi \left(-w_{\text{en}} u_t^2 - u \cdot \nabla V(u) - |u_x|_{\mathcal{D}}^2 + w_{\text{en}} \kappa |\mathcal{D} u_x \cdot u_t| + \frac{\kappa^2}{2} |u|_{\mathcal{D}}^2 \right) dx,$$

thus, polarizing the scalar product $\mathcal{D} u_x \cdot u_t$,

$$\partial_t \mathcal{F}(\xi, t) \leq \int_{\mathbb{R}} T_{\xi} \psi \left(\left(\frac{w_{\text{en}} \kappa^2 \lambda_{\mathcal{D}, \max}}{4} - 1 \right) |u_x|_{\mathcal{D}}^2 - u \cdot \nabla V(u) + \frac{\kappa^2 \lambda_{\mathcal{D}, \max}}{2} u^2 \right) dx,$$

and according to inequalities (18) satisfied by the quantity κ ,

$$(21) \quad \partial_t \mathcal{F}(\xi, t) \leq \int_{\mathbb{R}} T_{\xi} \psi \left(-\frac{|u_x|_{\mathcal{D}}^2}{2} - u \cdot \nabla V(u) + \frac{\lambda_{V, \min}}{4} u^2 \right) dx.$$

If the quantity $u(x, t)$ was close to $0_{\mathbb{R}^n}$ for all x in \mathbb{R} , then the right-hand side of this last inequality would be bounded from above by $-\varepsilon \mathcal{F}(\xi, t)$ for some positive quantity ε ; indeed, for $u(\cdot, \cdot)$ not larger than d_{Esc} , according to inequalities (13) derived from the definition of d_{Esc} , the last term is dominated by the term $-u \cdot \nabla V(u)$, and the quantities $u \cdot \nabla V(u)$ and u^2 and $V(u)$ do not differ by more than a bounded factor. What will actually follow (inequality (20) below) is indeed an upper bound of this form plus an additional term that comes from the part of space where $u(x, t)$ is *not* close to $0_{\mathbb{R}^n}$.

Let $\nu_{\mathcal{F}}$ be a positive quantity, sufficiently small so that

$$(22) \quad \nu_{\mathcal{F}} w_{\text{en}} \leq 1 \quad \text{and} \quad \nu_{\mathcal{F}} \left(w_{\text{en}} \lambda_{V, \max} + \frac{1}{2} \right) \leq \frac{\lambda_{V, \min}}{4}$$

(these two properties will be used in estimates below), namely

$$\nu_{\mathcal{F}} = \min \left(\frac{1}{w_{\text{en}}}, \frac{\lambda_{V, \min}}{4 w_{\text{en}} \lambda_{V, \max}} \right).$$

Let us add and subtract to the right-hand side of inequality (21) the same quantity (with the purpose of making appear a term proportional to $-\mathcal{F}(\xi, t)$), as follows:

$$(23) \quad \begin{aligned} \partial_t \mathcal{F}(\xi, t) &\leq \int_{\mathbb{R}} T_\xi \psi \left(-\frac{|u_x|_{\mathcal{D}}^2}{2} - \nu_{\mathcal{F}} \left(w_{\text{en}} V(u) + \frac{u^2}{2} \right) \right) dx \\ &\quad + \int_{\mathbb{R}} T_\xi \psi \left(\nu_{\mathcal{F}} \left(w_{\text{en}} V(u) + \frac{u^2}{2} \right) - u \cdot \nabla V(u) + \frac{\lambda_{V, \min}}{4} u^2 \right) dx. \end{aligned}$$

The following observations can be made about the right-hand side of inequality (23).

- According to the first of conditions (22) on $\nu_{\mathcal{F}}$, the first term is bounded from above by $-\nu_{\mathcal{F}} \mathcal{F}(\xi, t)$.
- According to estimates (13) on $V(u)$ and $u \cdot \nabla V(u)$ for $|u|$ not larger than d_{Esc} , and according to the choice (18) of κ (second condition) and to the choice (22) of $\nu_{\mathcal{F}}$ (second condition), the integrand of the second integral is nonpositive as long as x is *not* in $\Sigma_{\text{Esc}}(t)$. Therefore the inequality still holds if the domain of integration of the second integral is changed from \mathbb{R} to $\Sigma_{\text{Esc}}(t)$.

Finally, if we introduce the quantity:

$$K_{\mathcal{F}}(R) = \max_{|v| \leq R_{\max, \infty}(R)} \left(\nu_{\mathcal{F}} \left(w_{\text{en}} V(v) + \frac{v^2}{2} \right) - v \cdot \nabla V(v) + \frac{\lambda_{V, \min}}{4} v^2 \right),$$

(depending only on V and \mathcal{D} and R), then inequality (20) readily follows from (23). This finishes the proof of Lemma 2. \square

We are going to use the fact that the second term in expression (20) is small if the set $\Sigma_{\text{Esc}}(t)$ is “far away” from ξ .

4.3 Control of the distance to the minimum point by the firewall function

Let

$$(24) \quad d_{\text{esc}} = d_{\text{Esc}} \sqrt{\frac{\min\left(\frac{w_{\text{en}}}{2}, \frac{1}{4}\right)}{\max\left(\frac{1+\kappa}{2} \lambda_{\mathcal{D}, \max}, \frac{\lambda_{\mathcal{D}, \max}}{2}\right)}}.$$

As the quantity d_{Esc} defined in subsection 2.11, this quantity d_{esc} will provide a way to measure the vicinity of the solution to the minimum point $0_{\mathbb{R}^n}$, this time in terms of the firewall function \mathcal{F} . The value chosen for d_{esc} depends only on V and \mathcal{D} and ensures the validity of the following lemma.

Lemma 3 (escape/Escape). *For all ξ in \mathbb{R} and all t in $[0, +\infty)$, the following assertion holds:*

$$(25) \quad \mathcal{F}(\xi, t) \leq d_{\text{esc}}^2 \implies |u(\xi, t)|_{\mathcal{D}} \leq d_{\text{Esc}}.$$

Proof. Let v be a function in X . Then,

$$\begin{aligned}
|v(0)|_{\mathcal{D}}^2 &= \psi(0) |v(0)|_{\mathcal{D}}^2 \\
&\leq \frac{1}{2} \int_{\mathbb{R}} \left| \frac{d}{dx} (\psi(x) |v(x)|_{\mathcal{D}}^2) \right| dx \\
&\leq \frac{1}{2} \int_{\mathbb{R}} \left(|\psi'(x)| |v(x)|_{\mathcal{D}}^2 + 2\psi(x) |v(x) \cdot \mathcal{D}v'(x)| \right) dx \\
&\leq \frac{1}{2} \int_{\mathbb{R}} \psi(x) \left(\kappa \lambda_{\mathcal{D},\max} v(x)^2 + v(x)^2 + \lambda_{\mathcal{D},\max} |v'(x)|_{\mathcal{D}}^2 \right) dx \\
&\leq \max \left(\frac{1 + \kappa \lambda_{\mathcal{D},\max}}{2}, \frac{\lambda_{\mathcal{D},\max}}{2} \right) \int_{\mathbb{R}} \psi(x) (v(x)^2 + |v'(x)|_{\mathcal{D}}^2) dx.
\end{aligned}$$

It follows from inequality (19) on the coercivity of $\mathcal{F}(\cdot, \cdot)$ that, for all ξ in \mathbb{R} and t in $[0, +\infty)$,

$$|u(\xi, t)|_{\mathcal{D}}^2 \leq \frac{\max \left(\frac{1 + \kappa \lambda_{\mathcal{D},\max}}{2}, \frac{\lambda_{\mathcal{D},\max}}{2} \right)}{\min \left(\frac{w_{\text{en}}}{2}, \frac{1}{4} \right)} \mathcal{F}(\xi, t),$$

thus implication (25) holds with the value of d_{esc} chosen in (24). \square

4.4 Finite speed of invasion of the stable homogeneous equilibrium

The three next definitions ensure the validity of Lemma 4 below.

- Let $L(R)$ be a positive quantity, sufficiently large so that

$$(26) \quad K_{\mathcal{F}}(R) \frac{\exp(-\kappa L(R))}{\kappa} \leq \nu_{\mathcal{F}} \frac{d_{\text{esc}}^2}{8}, \quad \text{namely} \quad L(R) = \frac{1}{\kappa} \log \left(\frac{8 K_{\mathcal{F}}(R)}{\nu_{\mathcal{F}} d_{\text{esc}}^2 \kappa} \right).$$

- Let $\eta_{\text{no-esc}} : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ (“no-escape hull”) be the function defined by (see figure 9):

$$(27) \quad \eta_{\text{no-esc}}(x) = \begin{cases} +\infty & \text{for } x < 0, \\ \frac{d_{\text{esc}}^2}{2} \left(1 - \frac{x}{2L(R)} \right) & \text{for } 0 \leq x \leq L(R), \\ \frac{d_{\text{esc}}^2}{4} & \text{for } x \geq L(R). \end{cases}$$

- Let $c_{\text{no-esc}}(R)$ (“no-escape speed”) denote a positive quantity, sufficiently large so that

$$(28) \quad c_{\text{no-esc}}(R) \frac{d_{\text{esc}}^2}{4L(R)} \geq \frac{2K_{\mathcal{F}}}{\kappa}, \quad \text{namely} \quad c_{\text{no-esc}}(R) = \frac{8K_{\mathcal{F}}L(R)}{\kappa d_{\text{esc}}^2}.$$

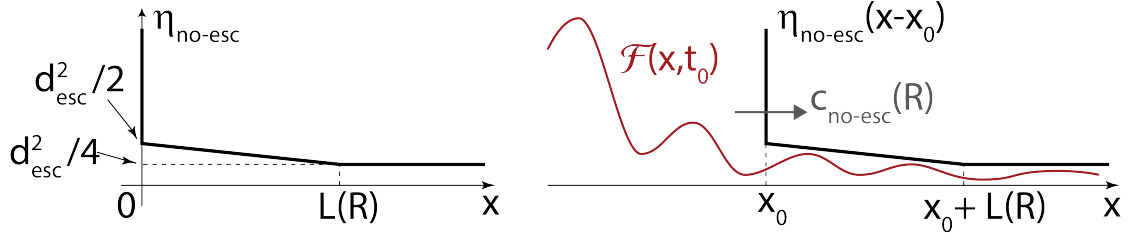


Figure 9: Left: graph of the hull function $\eta_{\text{no-esc}}$. Right: illustration of Lemma 4; if the firewall function is below the hull at a certain time and if the hulls travels to the right at speed $c_{\text{no-esc}}(R)$, then the firewall function will not catch again the travelling hull in the future.

The quantities $L(R)$ and $c_{\text{no-esc}}(R)$ and the hull function $\eta_{\text{no-esc}}$ all depend on V and \mathcal{D} and R (although the notation for the hull function does not make it apparent; note that this dependence was unclear in the author's previous paper [26], if not in his mind ☺).

The following lemma states that if the firewall function is dominated by a translate of the no-escape hull at a certain date, then it remains dominated in the future by the function defined by the no-escape hull travelling (to the right) at the no-escape speed (see figure 9).

Lemma 4 (bound on invasion speed). *For every x_0 in \mathbb{R} and t_0 in $[0, +\infty)$, if*

$$\mathcal{F}(x, t_0) \leq \eta_{\text{no-esc}}(x - x_0) \quad \text{for all } x \text{ in } \mathbb{R},$$

then, for every date t not smaller than t_0 ,

$$\mathcal{F}(x, t) \leq \eta_{\text{no-esc}}(x - x_0 - c_{\text{no-esc}}(R)(t - t_0)) \quad \text{for all } x \text{ in } \mathbb{R}.$$

Proof. Let x_0 in \mathbb{R} and t_0 in $[0, +\infty)$ such that

$$(29) \quad \mathcal{F}(x, t_0) \leq \eta_{\text{no-esc}}(x - x_0) \quad \text{for all } x \text{ in } \mathbb{R},$$

and let us consider the set $\mathcal{T}_{\text{no-esc}}$ (“no-escape times” between t_0 and $+\infty$) defined by

$$\mathcal{T}_{\text{no-esc}} = \{t \in [t_0, +\infty) : \mathcal{F}(x, t) \leq \eta_{\text{no-esc}}(x - x_0 - c_{\text{no-esc}}(R)(t - t_0)) \text{ for all } x \text{ in } \mathbb{R}\}.$$

According to inequality (29), the quantity t_0 belongs to this set. We are going to prove that the set $\mathcal{T}_{\text{no-esc}}$ is the whole interval $[t_0, +\infty)$ (this will prove the lemma since this assertion is equivalent to its conclusion).

Let δ be a positive quantity, sufficiently small so that

$$\delta \frac{2 K_{\mathcal{F}}(R)}{\kappa} \leq \frac{d_{\text{esc}}^2}{2}, \quad \text{namely: } \delta = \frac{\kappa d_{\text{esc}}^2}{4 K_{\mathcal{F}}(R)}.$$

The domain

$$\{(x, t) \in \mathbb{R}^2 : t_0 \leq t \leq t_0 + \delta\}$$

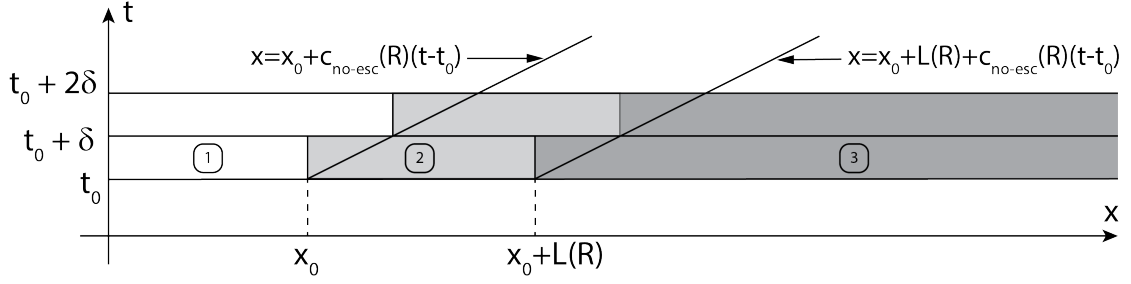


Figure 10: Areas “1”, “2”, and “3” in the domain $\mathbb{R} \times [t_0, t_0 + \delta]$ of the (x, t) -plane.

is shown on figure 10. It is divided into three areas labelled 1, 2, and 3.

According to the definition of $\eta_{\text{no-esc}}$, the quantity $\mathcal{F}(x, t_0)$ is bounded from above by $d_{\text{esc}}^2/2$ for all x not smaller than x_0 . On the other hand, according to estimate (20) on the time derivative of the firewall function, for all x in \mathbb{R} and all t in $[0, +\infty)$,

$$(30) \quad \partial_t \mathcal{F}(x, t) \leq \frac{2 K_{\mathcal{F}}(R)}{\kappa}.$$

Thus, according to the definition of δ , for all (x, t) in areas 2 and 3 of figure 10,

$$\mathcal{F}(x, t) \leq d_{\text{esc}}^2,$$

and it follows Lemma 3 that, for all t in $[t_0, t_0 + \delta]$,

$$\Sigma_{\text{Esc}}(t) \subset (-\infty, x_0).$$

Thus, for all (x, t) in area 3,

$$\int_{\Sigma_{\text{Esc}}(t)} T_x \psi(y) dy \leq \int_{-\infty}^{-L(R)} \psi(y) dy = \frac{\exp(-\kappa L(R))}{\kappa},$$

thus in view of the choice (26) of the quantity $L(R)$, estimate (20) on the time derivative of the firewall function yields, still for all (x, t) in area 3,

$$\partial_t \mathcal{F}(x, t) \leq -\nu_{\mathcal{F}} \mathcal{F}(x, t) + \frac{\nu_{\mathcal{F}} d_{\text{esc}}^2}{8}$$

and as a consequence, again for all (x, t) in area 3,

$$(31) \quad \mathcal{F}(x, t) > \frac{d_{\text{esc}}^2}{8} \implies \partial_t \mathcal{F}(x, t) < 0.$$

Thus, for all (x, t) in area 3, since according to the definition of $\eta_{\text{no-esc}}$ the quantity $\mathcal{F}(x, t_0)$ is bounded from above by $d_{\text{esc}}^2/4$, it follows from implication (31) that

$$(32) \quad \mathcal{F}(x, t) < d_{\text{esc}}^2/4.$$

On the other hand, for every x in $[x_0, x_0 + L(R)]$, the quantity

$$\Delta(x, t) = \mathcal{F}(x, t) - \frac{d_{\text{esc}}^2}{2} \left(1 - \frac{x - x_0 - c_{\text{no-esc}}(R)(t - t_0)}{2} \right)$$

has the following properties.

- At $t = t_0$, it equals:

$$\Delta(x, t_0) = \mathcal{F}(x, t_0) - \eta_{\text{no-esc}}(x - x_0)$$

and according to inequality (29) this quantity is nonpositive.

- According to the definition (28) of $c_{\text{no-esc}}(R)$ and to the upper bound (30) on $\partial_t \mathcal{F}$,

$$\partial_t \Delta(x, t) \leq 0 \quad \text{for all } t \text{ in } [0, +\infty).$$

It follows from these two properties that, for every (x, t) in area 2 of figure 10, $\Delta(x, t) \leq 0$, and as a consequence,

$$(33) \quad \mathcal{F}(x, t) \leq \eta_{\text{no-esc}}(x - x_0 - c_{\text{no-esc}}(R)(t - t_0)).$$

It follows from inequalities (32) and (33) that

$$[t_0, t_0 + \delta] \subset \mathcal{T}_{\text{no-esc}},$$

and the same reasoning can be repeated with t_0 replaced by $t_0 + \delta$ and x_0 replaced by $x_0 + c_{\text{no-esc}}(R)\delta$ (see figure 10). It follows that $\mathcal{T}_{\text{no-esc}}$ is equal to $[t_0, +\infty)$, and this proves Lemma 4. \square

4.5 Exponential convergence

Let us define the (asymptotic) “no-invasion” speed $c_{\text{no-inv}}$ as:

$$(34) \quad c_{\text{no-inv}} = c_{\text{no-esc}}(R_{\text{att}}) + 1$$

(this quantity depends only on V and \mathcal{D}).

Lemma 5 (exponential decrease of firewall). *There exist positive quantities $\tilde{\nu}_{\mathcal{F}}$ and $\tilde{K}_{\mathcal{F}}$, depending only on V and \mathcal{D} , such that, if there exist a real quantity x_0 and a nonnegative time t_0 satisfying:*

$$(35) \quad \sup_{t \geq t_0} \sup_{x \in \mathbb{R}} |u(x, t)| \leq R_{\text{att}} \quad \text{and} \quad \sup_{x \geq x_0} \mathcal{F}(x, t_0) \leq \frac{d_{\text{esc}}^2}{4},$$

then, for every time t not smaller than t_0 and every quantity x not smaller than $x_0 + c_{\text{no-inv}}(t - t_0)$,

$$(36) \quad \mathcal{F}(x, t) \leq \tilde{K}_{\mathcal{F}} \exp(-\tilde{\nu}_{\mathcal{F}}(t - t_0)).$$

Proof. Let $\eta_{\text{no-esc-att}}$ denote the function defined exactly as $\eta_{\text{no-esc}}$ in definition (27) but with the quantity $L(R_{\text{att}})$ instead of $L(R)$. This redefined “no escape hull” function is adapted to the first assumption of (35) above (uniform bound on the solution).

It follows from the second assumption of (35) that, for all x in \mathbb{R} ,

$$\mathcal{F}(x, t_0) \leq \eta_{\text{no-esc-att}}(x - x_0),$$

thus it follows from Lemma 4 and from the first assumption of (35) that, for all t larger than t_0 and all x in \mathbb{R} ,

$$\mathcal{F}(x, t) \leq \eta_{\text{no-esc-att}}(x - x_0 - c_{\text{no-esc}}(R_{\text{att}})(t - t_0)).$$

Then it follows from Lemma 3 and from the definition (27) of $\eta_{\text{no-esc}}$ that, for all t larger than t_0 ,

$$\Sigma_{\text{Esc}}(t) \subset (-\infty, x_0 + c_{\text{no-esc}}(R_{\text{att}})(t - t_0)]$$

(see figure 11). As a consequence, for all t in $[t_0, +\infty)$ and x in $[x_0 + c_{\text{no-inv}}(t - t_0), +\infty)$,

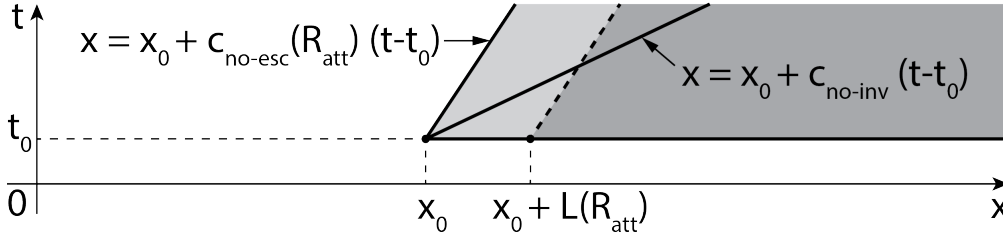


Figure 11: Illustration of Lemma 5. In the strong shaded area \mathcal{F} is not larger than $d_{\text{esc}}^2/4$, and in the light shaded area it is not larger than $d_{\text{esc}}^2/2$. Both shaded areas are disjoint from the “Escape” sets $\Sigma_{\text{Esc}}(t)$.

it follows from estimate (20) on $\partial_t \mathcal{F}$ and from the definition (34) of $c_{\text{no-inv}}$ that:

$$(37) \quad \partial_t \mathcal{F}(x, t) \leq -\nu_{\mathcal{F}} \mathcal{F}(x, t) + \frac{K_{\mathcal{F}}(R_{\text{att}})}{\kappa} \exp(-\kappa(t - t_0)).$$

Let us consider the quantity

$$\tilde{\nu}_{\mathcal{F}} = \min\left(\nu_{\mathcal{F}}, \frac{\kappa}{2}\right)$$

and let

$$\tilde{\mathcal{F}}(x, t) = \exp(\tilde{\nu}_{\mathcal{F}}(t - t_0)) \mathcal{F}(x, t).$$

It follows from inequality (37) above that, for all t in $[t_0, +\infty)$ and x in $[x_0 + c_{\text{no-inv}}(t - t_0), +\infty)$,

$$\partial_t \tilde{\mathcal{F}}(x, t) \leq \frac{K_{\mathcal{F}}(R_{\text{att}})}{\kappa} \exp\left(-\frac{\kappa}{2}(t - t_0)\right).$$

Thus, again for all t in $[t_0, +\infty)$ and x in $[x_0 + c_{\text{no-inv}}(t - t_0), +\infty)$,

$$(38) \quad \tilde{\mathcal{F}}(x, t) \leq \tilde{\mathcal{F}}(x, t_0) + \frac{2 K_{\mathcal{F}}(R_{\text{att}})}{\kappa^2}.$$

Thus, since $\tilde{\mathcal{F}}(x, t_0) = \mathcal{F}(x, t_0)$, if we consider the quantity

$$\tilde{K}_{\mathcal{F}} = \frac{d_{\text{esc}}^2}{4} + \frac{2 K_{\mathcal{F}}(R_{\text{att}})}{\kappa^2},$$

then inequality (36) follows from inequality (38) above. Lemma 5 is proved. \square

4.6 Proof of Proposition 3 (“sufficient condition for bistability”)

Lemma 6 (sufficient condition for small firewall at infinity). *There exists a positive quantity r , depending only on V and \mathcal{D} , such that, if*

$$(39) \quad \limsup_{x \rightarrow +\infty} \int_x^{x+1} (u_0(x)^2 + u_0'(x)^2) dx \leq r^2,$$

then there exists $x_{0,0}$ in \mathbb{R} such that, for all x larger than $x_{0,0}$,

$$(40) \quad \mathcal{F}(x, t) \leq \frac{d_{\text{esc}}^2}{4}.$$

Proof. This result follows readily from the definition of the firewall function \mathcal{F} . \square

Proof of Proposition 3. We are now in position to complete the proof of Proposition 3 on page 23. Let us assume that r is small enough so that Lemma 6 holds, let us assume that assumption (39) holds, and let $x_{0,0}$ be such that, for all x larger than $x_{0,0}$, inequality (40) holds.

According to Lemma 1, there exists a quantity $T_{\text{att}}(R)$ (“time to enter attracting ball”), depending only on V and \mathcal{D} and R , such that

$$(41) \quad \sup_{t \geq T_{\text{att}}(R)} \sup_{x \in \mathbb{R}} |u(x, t)| \leq R_{\text{att}}.$$

Lemma 4 can be applied twice (see figure 12).

- First between $t = 0$ and $t = T_{\text{att}}(R)$, for the speed $c_{\text{no-esc}}(R)$ and the quantity $L(R)$ both depending on the quantity R , starting from the property:

$$\mathcal{F}(x, 0) \leq \eta_{\text{no-esc}}(x - x_{0,0}) \quad \text{for all } x \text{ in } \mathbb{R}$$

for the hull function $\eta_{\text{no-esc}}$ corresponding to the quantity $L(R)$.

- Second, between $t = T_{\text{att}}(R)$ and $+\infty$ (once the attracting ball for the L^∞ -norm is reached), for the (slower) speed $c_{\text{no-esc}}(R_{\text{att}})$ and the (smaller) quantity $L(R_{\text{att}})$ now depending only on V and \mathcal{D} , not on R , starting from the property:

$$\mathcal{F}(x, T_{\text{att}}(R)) \leq \eta_{\text{no-esc}}(x - x_{0,0} - L(R) - c_{\text{no-esc}}(R) T_{\text{att}}(R)) \quad \text{for all } x \text{ in } \mathbb{R}$$

for the (different) hull function $\eta_{\text{no-esc}}$ defined by the quantity $L(R_{\text{att}})$.

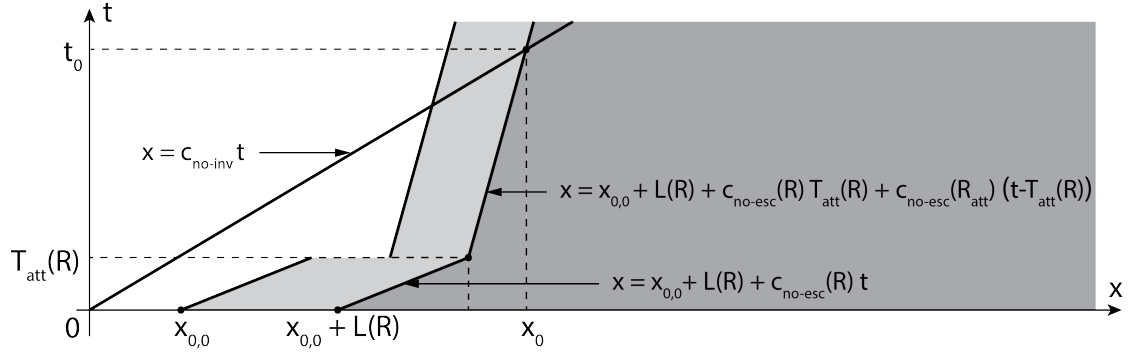


Figure 12: Illustration of the proof of Proposition 3. In the strong shaded area \mathcal{F} is not larger than $d_{\text{esc}}^2/4$, and in the light shaded area it is not larger than $d_{\text{esc}}^2/2$.

Let (x_0, t_0) in \mathbb{R}^2 be the solution of the following system (see figure 12):

$$\begin{cases} x_0 = c_{\text{no-inv}} t_0, \\ x_0 = x_{0,0} + L(R) + c_{\text{no-esc}}(R) T_{\text{att}}(R) + c_{\text{no-esc}}(R_{\text{att}}) (t_0 - T_{\text{att}}(R)), \end{cases}$$

that is

$$x_0 = c_{\text{no-inv}} t_0 \quad \text{and} \quad t_0 = x_{0,0} + L(R) + (c_{\text{no-esc}}(R) - c_{\text{no-esc}}(R_{\text{att}})) T_{\text{att}}(R).$$

According to Lemma 4 (applied twice as explained above), the hypotheses of Lemma 5 are satisfied for this definition of x_0 and t_0 . Proposition 3 follows from the conclusions of this lemma and the coercivity property (19) of the firewall function. \square

5 Asymptotic energy of a bistable solution

The aim of this section is to define the asymptotic energy of a bistable solution as stated in Proposition 1 on page 9. The fact that this asymptotic energy is either equal to minus infinity or nonnegative (completing the proof of Proposition 1) will be proved later, in subsection 6.4.

Assume that the potential V satisfies hypotheses (H_{coerc}) , (H_{norm}) , and (H_{min}) . Let (m_-, m_+) denote a pair of minimum points of V in the level set $V^{-1}(\{0\})$, let u_0 be a bistable initial condition connecting m_- to m_+ , and let $(x, t) \mapsto u(x, t)$ denote the corresponding solution for system (1).

Recall the quantity $c_{\text{no-inv}}$ (“no-invasion speed”) defined in Proposition 3. According to this proposition, both quantities

$$(42) \quad \sup_{x \leq -c_{\text{no-inv}} t} |u(x, t) - m_-| \quad \text{and} \quad \sup_{x \geq c_{\text{no-inv}} t} |u(x, t) - m_+|$$

approach 0 at an exponential rate when t approaches $+\infty$, and, according to the smoothing properties of the parabolic system (1) that provide the a priori bounds (10), the same

is true for the quantities

$$(43) \quad \sup_{|x| \geq c_{\text{no-inv}} t} |u_x(x, t)| \quad \text{and} \quad \sup_{|x| \geq c_{\text{no-inv}} t} |u_{xx}(x, t)| \quad \text{and} \quad \sup_{|x| \geq c_{\text{no-inv}} t} |u_t(x, t)|.$$

There are several ways to define the localized energy of the solution. The advantages of the following definition are that:

- it leads to natural estimates in terms of the firewall functionals defined in the previous section,
- it does not rely on the regularizing properties of system (1) — it is thus easier to extend to other classes of systems like the damped hyperbolic system (9),
- it provides the same explicit estimates as those that will be used for the proof of the upper semi-continuity of the asymptotic energy in section 8.

Let us consider the weight function χ defined by:

$$(44) \quad \chi(x, t) = \begin{cases} \exp(-\kappa(c_{\text{no-inv}} t - x)) = T_{-c_{\text{no-inv}} t} \psi(x) & \text{if } x \in (-\infty, c_{\text{no-inv}} t], \\ 1 & \text{if } x \in [-c_{\text{no-inv}} t, c_{\text{no-inv}} t], \\ \exp(-\kappa(x - c_{\text{no-inv}} t)) = T_{c_{\text{no-inv}} t} \psi(x) & \text{if } x \in [c_{\text{no-inv}} t, +\infty) \end{cases}$$

(see figure 13) and, for all $\xi \in \mathbb{R}$ and t in $[0, +\infty)$, let us consider the following quantities

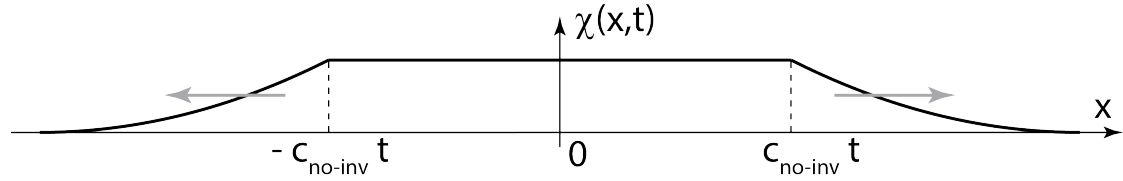


Figure 13: Graph of the weight function $x \mapsto \chi(x, t)$ defining the localized energy $\mathcal{E}(t)$.

(“localized energy” and “localized dissipation” respectively):

$$(45) \quad \begin{aligned} \mathcal{E}(t) &= \int_{\mathbb{R}} \chi(x, t) \left(\frac{|u_x(x, t)|_{\mathcal{D}}^2}{2} + V(u(x, t)) \right) dx, \\ \Delta(t) &= \int_{\mathbb{R}} \chi(x, t) u_t^2(x, t) dx. \end{aligned}$$

Lemma 7 (localized energy is almost decreasing). *There exists a positive quantity $K_{\mathcal{E}}$, depending only on V and \mathcal{D} , and a nonnegative time t_0 , depending only on V and \mathcal{D} and the solution under consideration, such that, for every time t not smaller than t_0 ,*

$$(46) \quad \mathcal{E}'(t) \leq -\frac{1}{2} \Delta(t) + K_{\mathcal{E}} \exp(-\tilde{\nu}_{\mathcal{F}}(t - t_0)).$$

Proof. For all $\xi \in \mathbb{R}$ and t in $[0, +\infty)$, let us consider the following quantities (“firewall functionals”):

$$(47) \quad \begin{aligned} \mathcal{F}_-(\xi, t) &= \int_{\mathbb{R}} T_{\xi} \psi(x) \left(w_{\text{en}} \left(\frac{|u_x(x, t)|_{\mathcal{D}}^2}{2} + V(u(x, t)) \right) + \frac{(u(x, t) - m_-)^2}{2} \right) dx, \\ \mathcal{F}_+(\xi, t) &= \int_{\mathbb{R}} T_{\xi} \psi(x) \left(w_{\text{en}} \left(\frac{|u_x(x, t)|_{\mathcal{D}}^2}{2} + V(u(x, t)) \right) + \frac{(u(x, t) - m_+)^2}{2} \right) dx, \end{aligned}$$

and let us consider the following subset of \mathbb{R} (the complement of the interval where $\chi(\cdot, t)$ is constant):

$$\Sigma_{\chi}(t) = (-\infty, -c_{\text{no-inv}} t) \cup (c_{\text{no-inv}} t, +\infty).$$

According to expression (11) of the derivative of a localized energy and since both quantities $\chi_x(x, t)$ and $\chi_t(x, t)$ vanish as soon as x is not in the set $\Sigma_{\chi}(t)$, for all t in $[0, +\infty)$,

$$\mathcal{E}'(t) = -\Delta(t) - \int_{\Sigma_{\chi}(t)} \chi_x \mathcal{D}u_x \cdot u_t dx + \int_{\Sigma_{\chi}(t)} \chi_t \left(\frac{|u_x(x, t)|_{\mathcal{D}}^2}{2} + V(u(x, t)) \right) dx.$$

Since

$$|\chi_x| \leq \kappa \chi \quad \text{and} \quad \chi_t(x, t) = \kappa c_{\text{no-inv}} \chi(x, t) \quad \text{for all } x \text{ in } \Sigma_{\chi}(t),$$

it follows that, still for all t in $[0, +\infty)$,

$$\mathcal{E}'(t) \leq -\frac{1}{2}\Delta(t) + \int_{\Sigma_{\chi}(t)} \chi \left(\kappa^2 \lambda_{\mathcal{D}, \max} \frac{|u_x(x, t)|_{\mathcal{D}}^2}{2} + \kappa c_{\text{no-inv}} \left(\frac{|u_x(x, t)|_{\mathcal{D}}^2}{2} + V(u(x, t)) \right) \right) dx.$$

Since according to (15) the quantities

$$w_{\text{en}} V(v) + \frac{(v - m_+)^2}{2} \quad \text{and} \quad w_{\text{en}} V(v) + \frac{(v - m_-)^2}{2}$$

are nonnegative for all v in \mathbb{R}^n , it follows that (observe the substitution of χ by $T_{-c_{\text{no-inv}} t} \psi$ and $T_{c_{\text{no-inv}} t} \psi$):

$$\begin{aligned} \mathcal{E}'(t) &\leq -\frac{1}{2}\Delta(t) \\ &+ \int_{-\infty}^{-c_{\text{no-inv}} t} T_{-c_{\text{no-inv}} t} \psi \left(\frac{\kappa^2 \lambda_{\mathcal{D}, \max}}{w_{\text{en}}} \left(w_{\text{en}} \left(\frac{|u_x|_{\mathcal{D}}^2}{2} + V(u) \right) + \frac{(v - m_-)^2}{2} \right) \right. \\ &+ \left. \frac{\kappa c_{\text{no-inv}}}{w_{\text{en}}} \left(w_{\text{en}} \left(\frac{|u_x|_{\mathcal{D}}^2}{2} + V(u) \right) + \frac{(v - m_-)^2}{2} \right) \right) dx \\ &+ \int_{c_{\text{no-inv}} t}^{+\infty} T_{c_{\text{no-inv}} t} \psi \left(\frac{\kappa^2 \lambda_{\mathcal{D}, \max}}{w_{\text{en}}} \left(w_{\text{en}} \left(\frac{|u_x|_{\mathcal{D}}^2}{2} + V(u) \right) + \frac{(v - m_+)^2}{2} \right) \right. \\ &+ \left. \frac{\kappa c_{\text{no-inv}}}{w_{\text{en}}} \left(w_{\text{en}} \left(\frac{|u_x|_{\mathcal{D}}^2}{2} + V(u) \right) + \frac{(v - m_+)^2}{2} \right) \right) dx; \end{aligned}$$

thus

$$(48) \quad \mathcal{E}'(t) \leq -\frac{1}{2}\Delta(t) + \frac{\kappa^2 \lambda_{\mathcal{D},\max} + \kappa c_{\text{no-inv}}}{w_{\text{en}}} (\mathcal{F}_-(-c_{\text{no-inv}} t, t) + \mathcal{F}_+(c_{\text{no-inv}} t, t)).$$

It follows from the same arguments as in subsection 4.6 on page 33 (proof of Proposition 3) that there exists t_0 in $[0, +\infty)$ such that, for all t not smaller than t_0 ,

$$\sup_{x \in \mathbb{R}} |u(x, t)| \leq R_{\text{att}} \quad \text{and} \quad \sup_{x \leq -c_{\text{no-inv}} t} \mathcal{F}_-(x, t) \leq \frac{d_{\text{esc}}^2}{4} \quad \text{and} \quad \sup_{x \geq c_{\text{no-inv}} t} \mathcal{F}_+(x, t) \leq \frac{d_{\text{esc}}^2}{4}.$$

As a consequence, according to Lemma 5 on page 31, for all t not smaller than t_0 ,

$$(49) \quad \sup_{x \leq -c_{\text{no-inv}} t} \mathcal{F}_-(x, t) \leq \tilde{K}_{\mathcal{F}} \exp(-\tilde{\nu}_{\mathcal{F}}(t - t_0))$$

$$\text{and} \quad \sup_{x \geq c_{\text{no-inv}} t} \mathcal{F}_+(x, t) \leq \tilde{K}_{\mathcal{F}} \exp(-\tilde{\nu}_{\mathcal{F}}(t - t_0)),$$

thus, if we consider the positive quantity

$$K_{\mathcal{E}} = \frac{2 \tilde{K}_{\mathcal{F}} (\kappa^2 \lambda_{\mathcal{D},\max} + \kappa c_{\text{no-inv}})}{w_{\text{en}}},$$

then inequality (46) follows from (48). Lemma 7 is proved. \square

Since the dissipation $\Delta(t)$ is nonnegative, it follows from inequality (46) of Lemma 7 that there exists a quantity

$$\mathcal{E}_{\infty}[u_0] \text{ in } \{-\infty\} \cup \mathbb{R}$$

such that

$$\mathcal{E}(t) \rightarrow \mathcal{E}_{\infty}[u_0] \quad \text{when} \quad t \rightarrow +\infty,$$

and, according (say) to the bounds (49) on the firewall functions and their coercivity property (19) on page 25, for all c in $[c_{\text{no-inv}}, +\infty)$, the quantity

$$\int_{-ct}^{ct} \left(\frac{|u_x(x, t)|_{\mathcal{D}}^2}{2} + V(u(x, t)) \right) dx$$

approaches the same limit $\mathcal{E}_{\infty}[u_0]$ when t approaches $+\infty$.

To complete the proof of Proposition 1, the last thing to prove is that this quantity $\mathcal{E}_{\infty}[u_0]$ is either equal to minus infinity or nonnegative, and this will be proved in subsection 6.4.

6 Relaxation of bistable solutions

The aim of this section is to prove Theorem 1.

For the whole section 6, let us assume that the potential V satisfies hypotheses (H_{coerc}) , (H_{norm}) , and (H_{min}) , let (m_-, m_+) denote a pair of minimum points of V in the level set

$V^{-1}(\{0\})$, let u_0 be a bistable initial condition connecting m_- to m_+ , and let $(x, t) \mapsto u(x, t)$ denote the corresponding solution for system (1). Assume in addition that the asymptotic energy $E_\infty[u_0]$ of this solution satisfies:

$$E_\infty[u_0] > -\infty.$$

To prove Theorem 1 amounts to prove that both quantities

$$(50) \quad \sup_{x \in \mathbb{R}} |u_t(x, t)| \quad \text{and} \quad \sup_{x \in \mathbb{R}} \text{dist} \left((u(x, t), u_x(x, t)), I(\mathcal{S}_{\text{bist}}(\mathcal{M}_0)) \right)$$

approach 0 when time approaches plus infinity.

6.1 Uniform approach to zero of the time derivative of the solution

The following lemma asserts the approach to 0 of the first among the two quantities in (50).

Lemma 8 (time derivative approaches zero). *The quantity*

$$\sup_{x \in \mathbb{R}} |u_t(x, t)|$$

approaches zero when time approaches plus infinity.

Proof. Let us keep the notation of the previous section 5. According to the approximate decrease of energy (46) and to the fact that $E_\infty[u_0]$ is not equal to $-\infty$, the nonnegative function $t \mapsto \Delta(t)$ is integrable on $[0, +\infty)$. For all t in $(0, +\infty)$,

$$(51) \quad \begin{aligned} \Delta'(t) &= \int_{\mathbb{R}} \left(\chi_t u_t^2 + 2 \chi u_t (-D^2 V(u) \cdot u_t + u_{xxt}) \right) dx, \\ &\leq \int_{\mathbb{R}} (c_{\text{no-inv}} \kappa \chi u_t^2 - 2 \chi u_t D^2 V(u) \cdot u_t - 2 \chi_x u_t u_{xt} - 2 \chi u_{xt}^2) dx. \end{aligned}$$

Let us consider the positive quantity

$$K_\Delta = c_{\text{no-inv}} \kappa - 2 \min_{|v| \leq R_{\text{att}}} \sigma(D^2 V(v)) + \frac{\kappa^2}{2}.$$

It follows from inequality (51) above that, for t sufficiently large:

$$(52) \quad \Delta'(t) \leq K_\Delta \Delta(t).$$

Since $t \mapsto \Delta(t)$ is nonnegative and integrable on $[0, +\infty)$, it follows that

$$\Delta(t) \rightarrow 0 \quad \text{when} \quad t \rightarrow +\infty.$$

It then follows from the a priori bounds (10) on the solution that

$$\sup_{|x| \leq c_{\text{no-inv}} t} |u_t(x, t)| \rightarrow 0 \quad \text{when} \quad t \rightarrow +\infty,$$

and it follows from assertion (43) about the behaviour of the solution outside of the interval $[-c_{\text{no-inv}}t, c_{\text{no-inv}}t]$ that

$$\sup_{x \in \mathbb{R}} |u_t(x, t)| \rightarrow 0 \quad \text{when} \quad t \rightarrow +\infty.$$

□

Remark. Using a compactness argument (on space and time intervals altogether), it is possible to derive Lemma 8 from the fact that $t \mapsto \Delta(t)$ is integrable on $[0, +\infty)$ without using inequality (52), see [12].

6.2 Compactness

The end of the proof of Theorem 1 will make an extensive use of the the following compactness argument.

Lemma 9 (compactness). *Let $(x_p, t_p)_{p \in \mathbb{N}}$ denote a sequence in $\mathbb{R} \times [0, +\infty)$ such that $t_p \rightarrow +\infty$ when $p \rightarrow +\infty$ and, for every integer p let us consider the functions $x \mapsto u_p(x)$ and $x \mapsto \tilde{u}_p(x)$ defined by:*

$$u_p(x) = u(x_p + x, t_p) \quad \text{and} \quad \tilde{u}_p(x) = u_t(x_p + x, t_p)$$

Then, up to replacing the sequence $(x_p, t_p)_{p \in \mathbb{N}}$ by a subsequence, there exists a stationary solution $x \mapsto u_\infty(x)$ in $\mathcal{C}_b^k(\mathbb{R}, \mathbb{R}^n)$ of system (1) such that, for every positive quantity L ,

$$\|u_p(\cdot) - u_\infty(\cdot)\|_{\mathcal{C}^k([-L, L], \mathbb{R}^n)} \rightarrow 0 \quad \text{and} \quad \|\tilde{u}_p(\cdot)\|_{\mathcal{C}^{k-2}([-L, L], \mathbb{R}^n)} \rightarrow 0 \quad \text{when} \quad p \rightarrow +\infty.$$

Proof. According to the a priori bounds (10) on the derivatives of the solutions of system (1), by compactness and a diagonal extraction procedure, there exist functions u_∞ and \tilde{u}_∞ such that, up to extracting a subsequence,

$$u_p(\cdot) \rightarrow u_\infty(\cdot) \quad \text{and} \quad \tilde{u}_p(\cdot) \rightarrow \tilde{u}_\infty \quad \text{when} \quad p \rightarrow +\infty,$$

uniformly on every compact subset of \mathbb{R} . The limits u_∞ and \tilde{u}_∞ belong respectively to $\mathcal{C}_b^k(\mathbb{R}, \mathbb{R}^n)$ and $\mathcal{C}_b^{k-2}(\mathbb{R}, \mathbb{R}^n)$ and the convergences hold in $\mathcal{C}^k([-L, L], \mathbb{R}^n)$ and in $\mathcal{C}^{k-2}([-L, L], \mathbb{R}^n)$ respectively, for every positive quantity L .

It follows from Lemma 8 that \tilde{u}_∞ vanishes identically, and passing to the limit in system (1) yields the conclusion that u_∞ is a stationary solution of this system. □

6.3 Approach to zero Hamiltonian level set for a sequence of times

It remains to prove that the second among the two quantities (50) also approaches 0 when time approaches $+\infty$. Recall the notation H (already defined in subsection 2.5) to denote the Hamiltonian associated to the differential system of stationary solutions of system (1):

$$H : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}, \quad (u, v) \mapsto \frac{|v|_{\mathcal{D}}^2}{2} - V(u).$$

Lemma 10 (approach to zero Hamiltonian level set for a sequence of times). *The following equality holds:*

$$(53) \quad \liminf_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}} |H(u(x, t), u_x(x, t))| = 0.$$

Proof. Let us proceed by contradiction and assume that the converse is true. Then there exists a positive quantity δ such that, for all sufficiently large positive t ,

$$(54) \quad \sup_{x \in \mathbb{R}} |H(u(x, t), u_x(x, t))| \geq \delta.$$

Observe that, for all x in \mathbb{R} and t in $[0, +\infty)$, the “space derivative of the Hamiltonian” along a solution has the following simple expression:

$$\partial_x \left(H(u(x, t), u_x(x, t)) \right) = u_x \cdot u_t.$$

In view of assertions (42) and (43) about the behaviour of the solution outside of the interval $[-c_{\text{no-inv}} t, c_{\text{no-inv}} t]$, hypothesis (54) yields:

$$\liminf_{t \rightarrow +\infty} \int_{-c_{\text{no-inv}} t}^{c_{\text{no-inv}} t} |u_x(x, t) \cdot u_t(x, t)| dx \geq 2\delta.$$

Thus, it follows from Hölder inequality and from the a priori bound (10) on $|u_x|$ that the limit

$$\liminf_{t \rightarrow +\infty} 2 c_{\text{no-inv}} t \int_{-c_{\text{no-inv}} t}^{c_{\text{no-inv}} t} u_t^2(x, t) dx$$

is positive. As a consequence the same is true for the limit

$$\liminf_{t \rightarrow +\infty} 2 c_{\text{no-inv}} t \Delta(t),$$

a contradiction with the fact that the function $t \mapsto \Delta(t)$ is integrable on $[1, +\infty)$. \square

6.4 Approach to zero Hamiltonian level set for all times

The aim of this subsection is to prove that the limit (53) of Lemma 10 holds for all time going to infinity, and not only for a subsequence of times (in other words that the \liminf in (53) can be substituted by a “full” limit). We will use the compactness Lemma 9 above and the results of subsection 12.2 about the value of the Lagrangian of stationary solutions. This goes through two lemmas.

As in subsection 2.5 and in section 12, let us consider the Hamiltonian and the (point-wise) Lagrangian associated to system (1):

$$H : \mathbb{R}^n \times \mathbb{R}^n, \quad (u, v) \mapsto \frac{|v|_{\mathcal{D}}^2}{2} - V(u) \quad \text{and} \quad L : \mathbb{R}^n \times \mathbb{R}^n, \quad (u, v) \mapsto \frac{|v|_{\mathcal{D}}^2}{2} + V(u).$$

The positive quantity δ_{Ham} defined in subsection 12.2 will also be used.

Lemma 11 (small Hamiltonian forces positive Lagrangian). *There exists a positive quantity T (depending on the solution $(x, t) \mapsto u(x, t)$ under consideration) such that, for every t larger than T and every x in \mathbb{R} ,*

$$|H(u(x, t), u_x(x, t))| \leq \delta_{\text{Ham}} \implies \int_x^{x+1} L(u(y, t), u_x(y, t)) dy \geq 0.$$

Proof. Let us proceed by contradiction and assume that the converse is true. Then there exists a sequence $(x_p, t_p)_{p \in \mathbb{N}}$ in $\mathbb{R} \times [0, +\infty)$ such that t_p approaches $+\infty$ when p approaches $+\infty$ and such that, for every integer p ,

$$(55) \quad |H(u(x_p, t_p), u_x(x_p, t_p))| \leq \delta_{\text{Ham}} \quad \text{and} \quad \int_{x_p}^{x_p+1} L(u(y, t), u_x(y, t)) dy < 0.$$

Up to extracting a subsequence, we may assume that the maps $x \mapsto u(x_p + x, t_p)$ converge, uniformly on every compact subset of \mathbb{R} , towards a stationary solution $x \mapsto u_\infty(x)$ of system (1), satisfying

$$(56) \quad |H(u_\infty(\cdot), u'_\infty(\cdot))| \leq \delta_{\text{Ham}} \quad \text{and} \quad \int_0^1 L(u_\infty(y), u'_\infty(y)) dy \leq 0.$$

According to Lemma 21 and to the first inequality of (56), there must exist a minimum point m of V in the level set $V^{-1}(\{0\})$ such that $|u_\infty(x)|_{\mathcal{D}} \leq d_{\text{Esc}}$ for all x in $[0, 1]$. Then it follows from the second inequality of (56) above that u_∞ must be identically equal to m , a contradiction with the second assertion of (55) above. \square

Lemma 12 (approach to zero Hamiltonian level set for all times). *The following limit holds:*

$$\sup_{x \in \mathbb{R}} |H(u(x, t), u_x(x, t))| \rightarrow 0 \quad \text{when} \quad t \rightarrow +\infty.$$

Proof. Let us proceed by contradiction and assume that the converse is true. Then, according to Lemma 10 and since the quantity

$$|H(u(x, t), u_x(x, t))|$$

depends continuously on x and t and is small for x and t large, there exists a positive quantity $\tilde{\delta}_{\text{Ham}}$, not larger than δ_{Ham} , and a sequence $(x_p, t_p)_{p \in \mathbb{N}}$ in $\mathbb{R} \times [0, +\infty)$ such that t_p approaches $+\infty$ when p approaches $+\infty$ and such that, for every integer p ,

$$|H(u(x_p, t_p), u_x(x_p, t_p))| = \tilde{\delta}_{\text{Ham}}.$$

Up to extracting a subsequence, we may assume that the maps $x \mapsto u(x_p + x, t_p)$ converge, uniformly on every compact subset of \mathbb{R} , towards a stationary solution $x \mapsto u_\infty(x)$ of system (1), satisfying

$$|H(u_\infty(\cdot), u'_\infty(\cdot))| = \tilde{\delta}_{\text{Ham}} \neq 0.$$

Since the Hamiltonian of this stationary solution is nonzero, this solution cannot be in $\mathcal{S}_{\text{bist}}(\mathcal{M}_0)$ and thus, according to Proposition 4 on page 62 (this is the key argument of this proof),

$$(57) \quad \int_{-\ell}^{\ell} L(u_{\infty}(y), u'_{\infty}(y)) dy \rightarrow +\infty \quad \text{when} \quad \ell \rightarrow +\infty.$$

Besides, it follows from assertions (42) and (43) about the behaviour of the solution outside of the interval $[-c_{\text{no-inv}}t, c_{\text{no-inv}}t]$ that, for p sufficiently large,

$$-c_{\text{no-inv}}t_p \leq x_p - \ell \quad \text{and} \quad x_p + \ell \leq c_{\text{no-inv}}t_p.$$

Thus, if Σ_p denotes the set

$$[-c_{\text{no-inv}}t_p, x_p - \ell] \cup [x_p + \ell, c_{\text{no-inv}}t_p],$$

then the energy $\mathcal{E}(t_p)$ defined in section 5 reads:

$$\int_{x_p - \ell}^{x_p + \ell} L(u(x, t_p), u_x(x, t_p)) dx + \int_{\Sigma_p} L(u(x, t_p), u_x(x, t_p)) dx.$$

According to Lemma 11 above, the second of these integrals is nonnegative, and according to the limit (57) above, the first of these integrals is positive and arbitrarily large if p is sufficiently large (depending on the choice of ℓ), a contradiction with the fact that the (almost decreasing) quantity $\mathcal{E}(t)$ is bounded from above uniformly with respect to t . \square

It follows from Lemma 11 and Lemma 12 that the asymptotic energy of the solution is nonnegative (provided that this asymptotic energy is not equal to minus infinity), and this finishes the proof of Proposition 1.

6.5 Approach to the set of zero Hamiltonian bistable stationary solutions

The following lemma completes the proof of Theorem 1.

Lemma 13 (approach to zero Hamiltonian bistable stationary solutions). *The following limit holds.*

$$\sup_{x \in \mathbb{R}} \text{dist} \left((u(x, t), u_x(x, t)), I(\mathcal{S}_{\text{bist}}(\mathcal{M}_0)) \right) \rightarrow 0 \quad \text{when} \quad t \rightarrow +\infty.$$

Proof. Let us proceed by contradiction and assume that the converse is true. Then there exists a positive quantity δ and a sequence $(x_p, t_p)_{p \in \mathbb{N}}$ in $\mathbb{R} \times [0, +\infty)$ such that t_p approaches $+\infty$ when p approaches $+\infty$ and such that, for every integer p ,

$$(58) \quad \text{dist} \left((u(x_p, t_p), u_x(x_p, t_p)), I(\mathcal{S}_{\text{bist}}(\mathcal{M}_0)) \right) \geq \delta.$$

Up to extracting a subsequence, we may assume that the maps $x \mapsto u(x_p + x, t_p)$ converge, uniformly on every compact subset of \mathbb{R} , towards a stationary solution $x \mapsto u_{\infty}(x)$

of system (1). According to Lemma 12, this stationary solution must have a zero Hamiltonian, and according to hypothesis (58) above, it cannot belong to the set $\mathcal{S}_{\text{bist}}(\mathcal{M}_0)$. As a consequence, according to Proposition 4 on page 62 (this is again the key argument of this proof),

$$\int_{-\ell}^{\ell} L(u_{\infty}(y), u'_{\infty}(y)) dy \rightarrow +\infty \quad \text{when} \quad \ell \rightarrow +\infty.$$

Thus, for ℓ sufficiently large, and for p sufficiently large (depending on the choice of ℓ), the quantity

$$\int_{x_p-\ell}^{x_p+\ell} L(u(x, t_p), u_x(x, t_p)) dx$$

is arbitrarily large, and the contradiction is the same as in the proof of Lemma 12 stated previously. \square

The proof of Theorem 1 is complete.

7 Convergence towards a standing terrace of bistable stationary solutions

Let us keep all the assumptions and notation of the previous section, and let us assume in addition that the potential V satisfies hypothesis (H_{disc}) , namely that the set $\mathcal{S}_{\text{bist}, \text{norm}}(\mathcal{M}_0)$ is totally disconnected in X . The aim of this section is to prove Theorem 2.

For all t in $[0, +\infty)$, let us consider the quantity $x_{\text{Esc}, 1, -}(t)$ in $\mathbb{R} \cup \{-\infty, +\infty\}$, defined as the infimum of the set

$$\{x \in \mathbb{R} : |u(x, t) - m_-|_{\mathcal{D}} = d_{\text{Esc}}\},$$

with the convention that the infimum equals $+\infty$ if this set is empty. It follows from assertion (42) on page 34 about the behaviour of the solution outside the interval between $-c_{\text{no-inv}}t$ and $c_{\text{no-inv}}t$ that, for every sufficiently large positive time t ,

$$(59) \quad \text{either} \quad x_{\text{Esc}, 1, -}(t) = +\infty \quad \text{or} \quad -c_{\text{no-inv}}t < x_{\text{Esc}, 1, -}(t) < c_{\text{no-inv}}t.$$

The end of the proof of Theorem 2 follows a standard pathway through the following series of (four) statements.

Lemma 14 (transversality at Escape point). *There exist positive quantities $\varepsilon_{\text{transv}}$ and T_{transv} such that, for every t in $[T_{\text{transv}}, +\infty)$, if $x_{\text{Esc}, 1, -}(t)$ is finite, then*

$$\left\langle u(x_{\text{Esc}, 1, -}(t), t) - m_-, u_x(x_{\text{Esc}, 1, -}(t), t) \right\rangle_{\mathcal{D}} \geq \varepsilon_{\text{transv}}.$$

Proof. Let us proceed by contradiction and assume that there exists a sequence $(t_p)_{p \in \mathbb{N}}$ such that t_p approaches $+\infty$ when p approaches $+\infty$ and such that, for every integer p ,

$$-\infty < x_{\text{Esc},1,-}(t_p) < +\infty \quad \text{and} \quad \left\langle u(x_{\text{Esc},1,-}(t_p), t_p) - m_-, u_x(x_{\text{Esc},1,-}(t_p), t_p) \right\rangle_{\mathcal{D}} \leq 1/p.$$

Up to extracting a subsequence, we may assume that the maps

$$x \mapsto u(x_{\text{Esc},1,-}(t_p) + x, t_p)$$

converge, uniformly on every compact subset of \mathbb{R} , towards a stationary solution $x \mapsto u_\infty(x)$ of system (1) satisfying

$$\langle u_\infty(0) - m_-, u'_\infty(0) \rangle_{\mathcal{D}} \leq 0 \quad \text{and, for all } x \text{ in } (-\infty, 0], \quad |u_\infty(x) - m_-|_{\mathcal{D}} \leq d_{\text{Esc}}$$

(the second property follows from the definition of $x_{\text{Esc},1,-}(t)$). This is contradictory to Lemma 19 on page 61. Lemma 14 is proved. \square

Corollary 2 (finiteness/infiniteness of $x_{\text{Esc},1,-}(\cdot)$). *One of the two following (mutually exclusive) alternatives occurs:*

1. *for every sufficiently large time t , the quantity $x_{\text{Esc},1,-}(t)$ equals $+\infty$,*
2. *(or) for every sufficiently large time t , the quantity $x_{\text{Esc},1,-}(t)$ is finite.*

In addition, if the second alternative occurs, then the map $t \mapsto x_{\text{Esc},1,-}(t)$ is of class (at least) C^1 on a neighbourhood of $+\infty$ and

$$x'_{\text{Esc},1,-}(t) \rightarrow 0 \quad \text{when} \quad t \rightarrow +\infty.$$

Proof. Let us consider the function

$$f : \mathbb{R}^n \times [0, +\infty) \rightarrow \mathbb{R}, \quad (x, t) \mapsto \frac{1}{2} (|u(x, t) - m_0|_{\mathcal{D}}^2 - d_{\text{Esc}}^2).$$

For all t in $[0, +\infty)$, if $x_{\text{Esc},1,-}(t)$ is finite then $f(x_{\text{Esc},1,-}(t), t) = 0$. If in addition t is not smaller than the quantity T_{transv} defined in Lemma 14, then

$$(60) \quad \partial_x f(x_{\text{Esc},1,-}(t), t) = \left\langle u(x_{\text{Esc},1,-}(t), t) - m_-, u_x(x_{\text{Esc},1,-}(t), t) \right\rangle_{\mathcal{D}} \geq \varepsilon_{\text{transv}} > 0.$$

Let us consider the set of values of t in $[T_{\text{transv}}, +\infty)$ such that $x_{\text{Esc},1,-}(t)$ is finite:

- it follows from inequality (60) and from the implicit function theorem that this set is open in $[T_{\text{transv}}, +\infty)$;
- it follows from the definition of $x_{\text{Esc},1,-}(t)$ and from assertion (59) about $x_{\text{Esc},1,-}(t)$ (up to replacing T_{transv} by a larger quantity we may assume that this assertion holds on $[T_{\text{transv}}, +\infty)$) that this set is closed in $[T_{\text{transv}}, +\infty)$.

As a consequence, this set is either empty or equal to $[T_{\text{transv}}, +\infty)$, and this proves the alternative (first assertion of the lemma).

From the same application of the implicit function theorem, it also follows (details are left to the reader) that, if the second alternative occurs, then the function $t \mapsto x_{\text{Esc},1,-}(t)$ is smooth on $[T_{\text{transv}}, +\infty)$ (it is as regular as the function f , thus at least of class \mathcal{C}^1). For every time t in this interval, the quantity $x'_{\text{Esc},1,-}(t)$ reads:

$$x'_{\text{Esc},1,-}(t) = -\frac{\partial_t f(x_{\text{Esc},1,-}(t), t)}{\partial_x f(x_{\text{Esc},1,-}(t), t)} = -\frac{\left\langle u(x_{\text{Esc},1,-}(t), t) - m_-, u_t(x_{\text{Esc},1,-}(t), t) \right\rangle_{\mathcal{D}}}{\left\langle u(x_{\text{Esc},1,-}(t), t) - m_-, u_x(x_{\text{Esc},1,-}(t), t) \right\rangle_{\mathcal{D}}}.$$

According to Lemma 8, the numerator of this expression approaches 0 when time approaches plus infinity, while according to inequality (60) the denominator remains larger than $\varepsilon_{\text{transv}}$; it follows that $x'_{\text{Esc},1,-}(t)$ approaches 0 when time approaches plus infinity. Corollary 2 is proved. \square

Lemma 15 (approach to a homogeneous equilibrium). *Assume that the first alternative of Corollary 2 occurs (that is, $x_{\text{Esc},1,-}(t)$ equals $+\infty$ for every sufficiently large time t). Then the local minimum points m_- and m_+ must be equal, and*

$$\sup_{x \in \mathbb{R}} |u(x, t) - m_{\pm}| \rightarrow 0 \quad \text{when } t \rightarrow +\infty.$$

Proof. The fact that $m_- = m_+$ follows from the definition of d_{Esc} . To prove uniform convergence towards $m_+ = m_-$, we may again proceed by contradiction and use a compactness argument, or use properties (19) and (20) on page 25 and on page 26 for the functional $t \mapsto \mathcal{F}(t)$. \square

The next lemma (and the repetition of the same argument if the number of “bumps” is larger than 1) is the only place in this paper where hypothesis (H_{disc}) is required.

Lemma 16 (approach to an inhomogeneous stationary solution). *Assume that the second alternative of Corollary 2 occurs (that is, $x_{\text{Esc},1,-}(t)$ is finite for every sufficiently large time t). Then there exists a stationary solution $u_{\infty,1}$ in the set $\mathcal{S}_{\text{bist, norm}}(\mathcal{M}_0)$ such that $u_{\infty,1}(x)$ approaches m_- when x approaches $-\infty$, and such that the maps*

$$\mathbb{R} \rightarrow \mathbb{R}^n, \quad x \mapsto u(x_{\text{Esc},1,-}(t) + x, t)$$

converge, uniformly on every compact subset of \mathbb{R} , towards $u_{\infty,1}$ when t approaches $+\infty$.

Proof. Take a sequence $(t_p)_{p \in \mathbb{N}}$, such that t_p approaches $+\infty$ when p approaches $+\infty$. Up to extracting a subsequence, we may assume that the maps

$$y \mapsto u(x_{\text{Esc},1,-}(t_p) + y, t_p)$$

converge, uniformly on every compact subset of \mathbb{R} , towards a stationary solution $y \mapsto u_{\infty,1}(y)$ of system (1). It follows from the definition of $x_{\text{Esc},1,-}(t)$ that

$$|u_{\infty,1}(0) - m_-|_{\mathcal{D}} = d_{\text{Esc}} \quad \text{and, for all } y \text{ in } (-\infty, 0), \quad |u_{\infty,1}(y) - m_-|_{\mathcal{D}} \leq d_{\text{esc}}.$$

Thus, it follows from Lemma 19 of subsection 12.1 that

$$u_{\infty,1}(y) \rightarrow m_- \quad \text{when} \quad y \rightarrow -\infty \quad \text{and, for all } y \text{ in } (-\infty, 0), \quad |u_{\infty,1}(y) - m_0|_{\mathcal{D}} < d_{\text{esc}},$$

and according to Lemma 13, this stationary solution $u_{\infty,1}$ must actually belong to $\mathcal{S}_{\text{bist, norm}}(\mathcal{M}_0)$.

Let \mathcal{L} denote the set of all possible limits (in the sense of uniform convergence on compact subsets of \mathbb{R}) of sequences of maps

$$x \mapsto u(x_{\text{Esc},1,-}(t'_p) + x, t'_p)$$

for all possible sequences $(t'_p)_{p \in \mathbb{N}}$ such that t'_p approaches $+\infty$ when p approaches $+\infty$. This set \mathcal{L} is included in $\mathcal{S}_{\text{bist, norm}}(\mathcal{M}_0)$, and, because the semi-flow of system (1) is continuous on X , this set \mathcal{L} is a continuum (a compact connected subset) of X .

Since on the other hand — according to hypothesis (H_{disc}) — the set $\mathcal{S}_{\text{bist, norm}}(\mathcal{M}_0)$ is totally disconnected, this set \mathcal{L} must actually be reduced to the singleton $\{u_{\infty,1}\}$. Lemma 16 is proved. \square

Let us assume that the second alternative of Corollary 2 occurs, that is $x_{\text{Esc},1,-}(t)$ is finite for t sufficiently large, and let us use the notation $u_{\infty,1}$ of this lemma. Let us denote by m_1 the limit of $u_{\infty,1}(x)$ when x approaches $+\infty$ (this point belongs to \mathcal{M}_0). Let L_1 denote the supremum of the (non empty) set

$$\{x \in \mathbb{R} : |u_{\infty,1}(x) - m_1|_{\mathcal{D}} = d_{\text{Esc}}\}.$$

According to Lemma 19 on page 61,

$$\langle u_{\infty,1}(L_1) - m_1, u'_{\infty,1}(L_1) \rangle_{\mathcal{D}} < 0.$$

As a consequence, for every sufficiently large time t there exists a unique quantity $x_{\text{Esc},1,+}(t)$ close to $x_{\text{Esc},1,-}(t) + L_1$ and such that

$$|u(x_{\text{Esc},1,+}(t), t) - m_1|_{\mathcal{D}} = d_{\text{Esc}}.$$

In addition, if we denote by $x_{\text{Esc},2,-}(t)$ the infimum of the set

$$\left\{ x \text{ in } (x_{\text{Esc},1,+}(t), +\infty) : |u(x, t) - m_1|_{\mathcal{D}} = d_{\text{Esc}} \right\}$$

(with the convention that $x_{\text{Esc},2,-}(t) = +\infty$ if this set is empty), then

$$x_{\text{Esc},2,-}(t) - x_{\text{Esc},1,+}(t) \rightarrow +\infty \quad \text{when} \quad t \rightarrow +\infty.$$

At this stage, it can be observed that Corollary 2 applies again, with $x_{\text{Esc},1,-}(t)$ replaced by $x_{2,-}(t)$ and m_- replaced by m_1 . Thus, there is again two cases alternative, depending on whether $x_{\text{Esc},2,-}(t)$ is finite or equals $+\infty$ when t is large.

1. If $x_{\text{Esc},2,-}(t)$ equals $+\infty$ for all t sufficiently large, then from similar arguments it follows that m_1 equals m_+ and

$$\sup_{x \in \mathbb{R}} |u(x, t) - u_{\infty,1}(x - x_{\text{Esc},1,-}(t))| \rightarrow 0 \quad \text{when } t \rightarrow +\infty,$$

and the conclusions of Theorem 2 hold with $q = 1$.

2. If on the other hand $x_{\text{Esc},2,-}(t)$ is finite for all t sufficiently large, then the procedure can be repeated again: it can be argued as in Lemma 16 that there exists $u_{\infty,2}$ in $\mathcal{S}_{\text{bist}, \text{norm}}(\mathcal{M}_0)$ such that the solution converges towards $u_{\infty,2}$ around $x_{\text{Esc},2,-}(t)$. And the scheme can be repeated introducing the infimum $x_{3,-}(t)$ of the set

$$\left\{ x \text{ in } (x_{\text{Esc},2,+}(t), +\infty) : |u(x, t) - m_2|_{\mathcal{D}} \geq d_{\text{Esc}} \right\} \quad \text{where } m_2 = \lim_{x \rightarrow +\infty} u_{\infty,2}(x)$$

and $x_{2,+}(t)$ is defined the same way as $x_{1,+}(t)$ above.

Because the localized energy $\mathcal{E}(t)$ is bounded from above, the procedure must eventually end up for some q in \mathbb{N}^* for which $x_{\text{Esc},q+1,-}(t)$ equals $+\infty$ for all t sufficiently large. Then the limit m_q at $+\infty$ of the last stationary solution $u_{\infty,q}$ must be equal to m_+ , and convergence of the solution towards the standing terrace of Theorem 2 follows. This finishes the proof of Theorem 2.

8 Upper semi-continuity of the asymptotic energy

The aim of this section is to prove Proposition 2 about the upper semi-continuity of the asymptotic energy with respect to bistable initial data.

Let us assume that V satisfies hypotheses (H_{coerc}) , (H_{norm}) , and (H_{min}) , let (m_-, m_+) denote a pair of minimum points of V in the level set $V^{-1}(\{0\})$, let $(u_{0,p})_{p \in \mathbb{N}}$ denote a sequence of functions in $X_{\text{bist}}(m_-, m_+)$ (bistable initial conditions connecting m_- to m_+), and let $u_{0,\infty}$ denote a function in $X_{\text{bist}}(m_-, m_+)$, such that

$$\|u_{0,p} - u_{0,\infty}\|_X \rightarrow 0 \quad \text{when } p \rightarrow +\infty.$$

Our aim is to prove that

$$\mathcal{E}_{\infty}[u_{\infty,0}] \geq \limsup_{p \rightarrow +\infty} \mathcal{E}_{\infty}[u_{p,0}].$$

For every p in $\mathbb{N} \cup \{+\infty\}$ and for all x in \mathbb{R} and t in $[0, +\infty)$, let $u_p(x, t) = (S_t u_{p,0})(x)$ denote the solution of system (1) with initial data $u_{p,0}$. Let us consider the same weight function $(x, t) \mapsto \chi(x, t)$ as the one defined in (44) on page 35, and, for every p in $\mathbb{N} \cup \{+\infty\}$ and for all ξ in \mathbb{R} , let us consider the quantities

$$\mathcal{E}_p(t) \quad \text{and} \quad \mathcal{F}_{+,p}(t) \quad \text{and} \quad \mathcal{F}_{-,p}(t)$$

defined exactly as in (45) and (47) on page 35 and on page 36 for the solution u_p .

Lemma 17 (uniform bound on the derivative of localized energies). *There exists a nonnegative time t_0 and an integer p_0 such that, for every integer p larger than p_0 and every time t larger than t_0 , the following inequality holds:*

$$(61) \quad \mathcal{E}'_p(t) \leq K_{\mathcal{E}} \exp(-\tilde{\nu}_{\mathcal{F}}(t - t_0)).$$

Proof. Inequality (61) will follow from inequality (46) on page 35 (the sole additional requirement is some uniformity with respect to p). Let R denote the supremum of the set

$$\{ \|u_{0,p}\|_X : p \in \mathbb{N} \cup \{+\infty\} \}$$

(this quantity is finite). According to Lemma 1 on page 20, there exists a quantity $T_{\text{att}}(R)$ (depending on V and \mathcal{D} and R , but not on p), such that, for every quantity t larger than $T_{\text{att}}(R)$ and every p in $\mathbb{N} \cup \{\infty\}$,

$$\sup_{x \in \mathbb{R}} |u_p(x, t)| \leq R_{\text{att}}.$$

It follows from the same arguments as in subsection 4.6 on page 33 (proof of Proposition 3) that there exists a time t_0 not smaller than $T_{\text{att}}(R)$ such that

$$\sup_{\xi \leq -c_{\text{no-inv}} t_0} \mathcal{F}_{-, \infty}(\xi, t_0) \leq \frac{d_{\text{esc}}^2}{8} \quad \text{and} \quad \sup_{\xi \geq c_{\text{no-inv}} t_0} \mathcal{F}_{+, \infty}(\xi, t_0) \leq \frac{d_{\text{esc}}^2}{8}.$$

Then, by continuity of the semi-flow in X , there exists an integer p_0 such that, for every integer p larger than p_0 ,

$$\sup_{\xi \leq -c_{\text{no-inv}} t_0} \mathcal{F}_{-, p}(\xi, t_0) \leq \frac{d_{\text{esc}}^2}{4} \quad \text{and} \quad \sup_{\xi \geq c_{\text{no-inv}} t_0} \mathcal{F}_{+, p}(\xi, t_0) \leq \frac{d_{\text{esc}}^2}{4}.$$

Then it follows from Lemma 5 on page 31 that, for every integer p larger than p_0 and for all t larger than t_0 ,

$$\mathcal{F}_{-, p}(-c_{\text{no-inv}} t, t) \leq \tilde{K}_{\mathcal{F}} \exp(-\tilde{\nu}_{\mathcal{F}}(t - t_0)) \quad \text{and} \quad \mathcal{F}_{+, p}(c_{\text{no-inv}} t, t) \leq \tilde{K}_{\mathcal{F}} \exp(-\tilde{\nu}_{\mathcal{F}}(t - t_0)).$$

Inequality (61) thus follows from inequality (46). Lemma 17 is proved. \square

Since $\mathcal{E}_p(t)$ approaches $\mathcal{E}_{\infty}[u_{p,0}]$ when t approaches $+\infty$, it follows from inequality (61) of Lemma 17 that, still for every integer p larger than p_0 and for all t larger than t_0 ,

$$\mathcal{E}_p(t) \geq \mathcal{E}_{\infty}[u_{p,0}] - \frac{K_{\mathcal{E}}}{\tilde{\nu}_{\mathcal{F}}} \exp(-\tilde{\nu}_{\mathcal{F}}(t - t_0)).$$

Passing to the limit as p approaches $+\infty$, it follows from the continuity of the semi-flow in X that, for all t larger than t_0 ,

$$\mathcal{E}_{\infty}(t) \geq \limsup_{p \rightarrow +\infty} \mathcal{E}_{\infty}[u_{p,0}] - \frac{K_{\mathcal{E}}}{\tilde{\nu}_{\mathcal{F}}} \exp(-\tilde{\nu}_{\mathcal{F}}(t - t_0)).$$

Finally, passing to the limit as t approaches $+\infty$, it follows that

$$\mathcal{E}_{\infty}[u_{\infty,0}] \geq \limsup_{p \rightarrow +\infty} \mathcal{E}_{\infty}[u_{p,0}],$$

which was the desired result. The proof of Proposition 2 is complete.

9 Existence results for stationary solutions and basin of attraction of a stable homogeneous solution

The aim of this section is to recover standard results concerning existence of homoclinic or heteroclinic stationary solutions and the basin of attraction of a stable homogeneous solution, as direct consequences of Theorem 1 on page 9 and Proposition 2 on page 10 (upper semi-continuity of the asymptotic energy). These results are stated as four independent corollaries. The proofs are given after the four statements. Elementary examples illustrating these results will be discussed in the next section.

9.1 Existence results for stationary solutions

The following two corollaries deal with the stationary solutions of system (1), and are variants of well-known results, usually obtained by calculus of variation techniques (minimization or mountain-pass arguments, see references below).

9.1.1 Case where the potential takes only nonnegative values

The following “minimization” corollary is illustrated by cases (a) and (b) of figure 2 on page 9. It is similar to (or contained in) results going back to the early nineties (see P. Rabinowitz [23] and P. Sternberg [31] and for instance N. Alikakos and G. Fusco [1] for recent results and additional references). It is by the way implicitly contained in Theorem 3 of Béthuel, Orlandi, Smets [4].

Let $\text{card}(\mathcal{M}_0)$ denote the cardinal of the set \mathcal{M}_0 .

Corollary 3 (existence of a chain of heteroclinic stationary solutions). *Assume that V satisfies hypotheses (H_{coerc}) , (H_{norm}) , and (H_{min}) . Assume furthermore that:*

- *the potential V takes only nonnegative values,*
- *and the number of minimum points of V in the level set $V^{-1}(\{0\})$ is larger than 1 (in other words $\text{card}(\mathcal{M}_0)$ is larger than 1).*

Then, for every pair (m_-, m_+) in \mathcal{M}_0^2 such that m_- differs from m_+ , there exist a nonzero integer q and $q - 1$ distinct minimum points m_1, \dots, m_{q-1} in \mathcal{M}_0 such that, if m_- is denoted by m_0 and m_+ by m_q , then for every integer i in $\{0, \dots, q - 1\}$, the set $\mathcal{S}_{\text{bist}}(m_i, m_{i+1})$ is nonempty. In other words, there exists a “chain” of bistable stationary solutions connecting m_- to m_+ .

9.1.2 Case where the potential takes negative values

The following “mountain pass” corollary is illustrated by cases (c), (d), and (e) of figure 2 on page 9. It is similar to (or contained in) results going back the early nineties (see A. Ambrosetti and M. L. Bertotti [2], Bertotti [3], and Rabinowitz and K. Tanaka [24]).

Corollary 4 (existence of a homoclinic stationary solution). *Assume that V satisfies hypotheses (H_{coerc}) , (H_{norm}) , and (H_{min}) . Assume furthermore that:*

- the potential V takes negative values,
- and there exists exactly one minimum point of V in the level set $V^{-1}(\{0\})$ (in other words \mathcal{M}_0 is reduced to a singleton).

Then there exists at least one nonconstant stationary solution that is homoclinic to the sole point in \mathcal{M}_0 . In other words, if m denotes this point, the set $W^u(m, 0) \cap W^s(m, 0)$ is nonempty.

9.2 Basin of attraction of a stable homogeneous stationary solution

The next two corollaries can be viewed as “dynamical” versions of the two previous ones. They require the following notation.

Notation. If m is a minimum point of V in the zero level set, let $\mathcal{B}_{\text{att}}(m)$ denote the basin of attraction (for the semi-flow of system (1)) of the homogeneous equilibrium m , that is:

$$\mathcal{B}_{\text{att}}(m) = \{u_0 \in X : (S_t u_0)(x) \rightarrow m, \text{ uniformly with respect to } x, \text{ when } t \rightarrow +\infty\},$$

and let $\partial\mathcal{B}_{\text{att}}(m)$ denote the topological border, in X , of $\mathcal{B}_{\text{att}}(m)$.

9.2.1 Case where the potential takes only nonnegative values

Corollary 5 below applies to example (c) of figure 2 on page 9, modified so that the global minimum value of V is 0. As Corollary 3 above, it is implicitly contained in Theorem 3 of Béthuel, Orlandi, Smets [4].

Corollary 5 (global stability of the unique global minimum point). *Assume that V satisfies hypotheses (H_{coerc}) , (H_{norm}) , and (H_{min}) . Assume furthermore that:*

- the potential V takes only nonnegative values,
- and the set \mathcal{M}_0 is reduced to a single point m (which is therefore the unique global minimum point of V),
- and there exists no nonconstant stationary solution homoclinic to m (in other words the set $\mathcal{S}_{\text{bist}}(m, m)$ is reduced to the function identically equal to m).

Then every bistable solution connecting m to m converges to m , uniformly in space, when time approaches infinity. In other words,

$$X_{\text{bist}}(m, m) = \mathcal{B}_{\text{att}}(m).$$

9.2.2 Case where the potential takes negative values

Corollary 6 below applies to cases (c), (d), and (e) of figure 2 on page 9, and is analogous in spirit to results of author's previous paper [25]. It is somehow related to the huge amount of existing literature about (codimension one) threshold phenomena in reaction-diffusion equations, going back (at least) to Fife's paper [10] of 1979 and the contributions of G. Flores in the late eighties [11]. Other references about this subject can be found in the recent paper [19] of Muratov and Zhong, where various threshold results of the same kind are obtained. The arguments used by these authors are based on the energy functional (2) on page 2, and are quite close in essence (although applied in a different setting limited to the scalar case $n = 1$) to those of the present paper and of the companion paper [27].

Corollary 6 (attractor of the border of the basin of attraction of a local minimum point). *Assume that V satisfies hypotheses (H_{coerc}) , (H_{norm}) , and (H_{min}) . Assume furthermore that the potential V takes negative values. Then, for every minimum point m in the level set $V^{-1}(\{0\})$, the following conclusions hold.*

- *There exists at least one bistable initial condition connecting m to himself and belonging to the border of the basin of attraction of the spatially homogeneous equilibrium m . In other words:*

$$\partial\mathcal{B}_{\text{att}}(m) \cap X_{\text{bist}}(m, m) \neq \emptyset.$$

- *Every bistable initial condition in this nonempty set has a positive asymptotic energy. As a consequence, as in Theorem 1, a solution $(x, t) \mapsto u(x, t)$ of system (1) with initial data in this set converges towards the set $\mathcal{S}_{\text{bist}}(\mathcal{M}_0)$ in the sense that both quantities:*

$$\sup_{x \in \mathbb{R}} |u_t(x, t)| \quad \text{and} \quad \sup_{x \in \mathbb{R}} \text{dist} \left(\left(u(x, t), u_x(x, t) \right), I(\mathcal{S}_{\text{bist}}(\mathcal{M}_0)) \right)$$

approach 0 when time approaches infinity.

Remark. Assume that the potential V takes only nonnegative values, and has a unique global minimum point m (the two first among the three hypotheses of Corollary 5).

- If furthermore n equals 1 (scalar case), then the set $\mathcal{S}_{\text{bist}}(m, m)$ is necessarily empty (indeed every solution of the Hamiltonian system (5) on page 7 in the unstable manifold of $(m, 0)$ must approach infinity when time approaches infinity, since the velocity can never vanish). As a consequence, the conclusions of Corollary 5 hold: every bistable solution connecting m to m converges to m , uniformly with respect to the space coordinate, when time approaches infinity.
- The situation is quite different in the vector case $n > 1$, where nonconstant stationary solutions homoclinic to a unique global minimum point might very well exist. Here is an example (the parameter ε is a small positive quantity):

$$V : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad (u_1, u_2) \mapsto -\frac{u_1^2 + u_2^2}{2} + \frac{(u_1^2 + u_2^2)^2}{4} - \varepsilon u_1.$$

For additional information and comments see P. Coullet [5].

9.3 Proof of Corollaries 3 and 5 (the potential takes only nonnegative values)

Let us assume that V satisfies hypotheses (H_{coerc}) , (H_{norm}) , and (H_{min}) , and that it takes only nonnegative values, thus:

$$\min_{u \in \mathbb{R}^n} V(u) = 0.$$

In this case the asymptotic energy of every bistable initial condition in $X_{\text{bist}}(\mathcal{M}_0)$ is nonnegative (since it is a limit of nonnegative quantities), therefore every bistable solution must converge towards the set $I(\mathcal{S}_{\text{bist}}(\mathcal{M}_0))$, as stated in Theorem 1.

Let us assume that the set \mathcal{M}_0 of (global) minimum points of V in level set $V^{-1}(\{0\})$ is not reduced to a singleton, and let m_- and m_+ be two distinct points in this set. We know from Corollary 1 that the set $X_{\text{bist}}(m_-, m_+)$ of bistable initial conditions connecting these two points is nonempty. If u_0 is a function (an initial condition) in this set, then the conclusions of Theorem 1 show that the set $I(\mathcal{S}_{\text{bist}}(\mathcal{M}_0))$ must connect the two points $(m_-, 0)$ and $(m_+, 0)$ in \mathbb{R}^{2n} . This proves that there exists a “chain” of heteroclinic stationary solutions connecting m_- to m_+ . Corollary 3 is thus proved.

Let us assume conversely that the set \mathcal{M}_0 is reduced to a single point m and that there is no nonconstant stationary solution homoclinic to m . Then the set $I(\mathcal{S}_{\text{bist}}(\mathcal{M}_0))$ is reduced to the singleton $\{(m, 0)\}$, and the conclusions of Theorem 1 show that a bistable solution connecting m to himself must converge to m , uniformly in space, when time approaches infinity. This proves Corollary 5.

9.4 Proof of Corollaries 4 and 6 (the potential takes negative values)

Let us assume that V satisfies hypotheses (H_{coerc}) , (H_{norm}) , and (H_{min}) , and that it takes negative values. Let m_0 be a minimum point of V in the level set $V^{-1}(\{0\})$, and let u_{neg} be a point of \mathbb{R}^n where the value of V is negative.

We are going to build a one-parameter family of bistable initial conditions in the set $X_{\text{bist}}(m_0, m_0)$, connecting (at both ends of the parameter range) the spatially homogeneous equilibrium m_0 to a bistable initial condition having negative energy. Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be a smooth cut-off function satisfying:

$$\chi(x) = 1 \text{ for all } x \text{ in } (-\infty, 0] \text{ and } \chi(x) = 0 \text{ for all } x \text{ in } [1 + \infty).$$

Let L be a (large) positive quantity, and let $u_{0,1}$ be the bistable initial condition connecting m_0 to himself defined by:

$$u_{0,1}(x) = \begin{cases} m_0 + \chi(x - L)(u_{\text{neg}} - m_0) & \text{for } x \geq 0 \\ u_{0,1}(-x) & \text{for } x \leq 0 \end{cases}$$

(see figure 14). Since $u_{0,1}(x) = u_{\text{neg}}$ for all x in $[-L, L]$ and since $V(u_{\text{neg}})$ is negative, the quantity L can be chosen large enough so that the (initial) energy

$$\int_{-\infty}^{+\infty} \left(\frac{|u'_{0,1}(x)|^2}{2} + V(u_{0,1}(x)) \right) dx$$

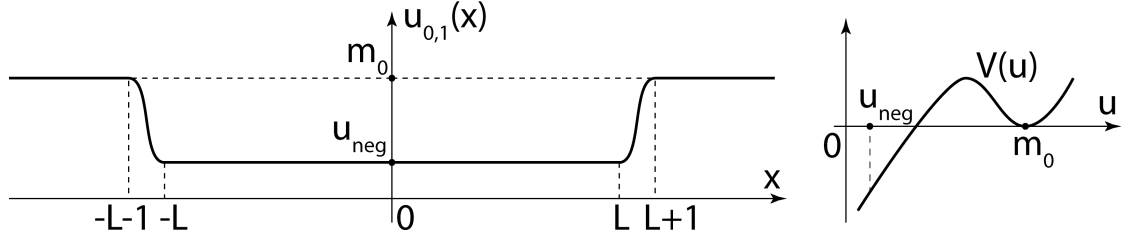


Figure 14: Graph of the function $x \mapsto u_{0,1}(x)$.

is negative. In this case it follows from Proposition 3 on page 23 that $u_{0,1}$ belong to $X_{\text{bist}}(m_0, m_0)$ (it is a bistable initial condition connecting m_0 to himself), and it follows from the expression (3) on page 2 of the derivative of the (non localized) energy that its asymptotic energy cannot be nonnegative. As a consequence it cannot belong to the basin of attraction $\mathcal{B}_{\text{att}}(m_0)$ of the homogeneous equilibrium m_0 . Now let us consider the one-parameter family $(u_{0,s})_{s \in [0,1]}$ of bistable initial conditions in $X_{\text{bist}}(m_0, m_0)$ defined by:

$$u_{0,s} = (1 - s)m_0 + s(u_{0,1} - m_0).$$

Then $u_0 \equiv m_0$, thus u_0 belong to $\mathcal{B}_{\text{att}}(m_0)$, and $u_{0,1}$ does not. It follows that there must exist $s_{\text{thres}} \in (0, 1]$ such that $u_{0,s_{\text{thres}}} \in \partial\mathcal{B}_{\text{att}}(m_0)$, and as a consequence that the set $X_{\text{bist}}(m_0, m_0) \cap \partial\mathcal{B}_{\text{att}}(m_0)$ is non empty.

On the other hand, since the asymptotic energy is upper semi-continuous (Proposition 2 on page 10), every initial condition in $\partial\mathcal{B}_{\text{att}}(m_0)$ must have a nonnegative asymptotic energy. More accurately, according to Lemma 21 on page 63, every initial condition in $\partial\mathcal{B}_{\text{att}}(m_0)$ must have a positive asymptotic energy. This proves Corollary 6.

As stated in Theorem 1, every solution in $\partial\mathcal{B}_{\text{att}}(m_0)$ must then approach the set $I(\mathcal{S}_{\text{bist}}(\mathcal{M}_0))$ as time approaches infinity. It follows that this set is not reduced to the point $(m_0, 0)$, or else such a solution would approach m_0 uniformly in space and thus belong to the basin of attraction $\mathcal{B}_{\text{att}}(m_0)$ and not its border, a contradiction. If moreover the set \mathcal{M}_0 of minimum points of V in level set $V^{-1}(\{0\})$ is reduced to the singleton $\{m_0\}$, then it follows that there exists at least one nonconstant stationary solution that is homoclinic to m_0 , and this proves Corollary 4.

10 Examples

This section is devoted to a discussion on elementary examples in the scalar case (the state variable $u(x, t)$ belongs to \mathbb{R}), corresponding to the potentials illustrated on figure 2 on page 9.

10.1 Allen–Cahn equation

The equation reads (see example (a) of figure 2):

$$u_t = u - u^3 + u_{xx} = -V'(u) + u_{xx} \quad \text{where} \quad V(u) = -u^2/2 + u^4/4 + 1/4.$$

In this example the set \mathcal{M}_0 is made of the two points -1 and 1 , and the set $\mathcal{S}_{\text{bist}}(\mathcal{M}_0)$ consists of:

- the “kink” solution $x \mapsto \tanh(x/\sqrt{2})$,
- and the “antikink” solution $x \mapsto -\tanh(x/\sqrt{2})$

(and their translates with respect to x).

Hypotheses (H_{coerc}) , (H_{norm}) , (H_{min}) , and (H_{disc}) are satisfied, and, according to Theorem 2, for every initial condition u_0 in $X_{\text{bist}}(\pm 1, \pm 1)$, the solution $S_t u_0$ approaches, when t approaches $+\infty$, a standing terrace involving a finite number of alternatively kink and antikink solutions, getting slowly away from one another.

Since the long-range interaction between two consecutive kink and antikink solutions is attractive, the following more precise result actually holds. In the sentences below, “approaches” means “approaches when t approaches $+\infty$, uniformly with respect to x in \mathbb{R} ”.

- If u_0 is in $X_{\text{bist}}(-1, -1)$, then $S_t u_0$ approaches -1 .
- If u_0 is in $X_{\text{bist}}(+1, +1)$, then $S_t u_0$ approaches $+1$.
- If u_0 is in $X_{\text{bist}}(-1, +1)$, then there exists $x_0 \in \mathbb{R}$ such that $S_t u_0$ approaches the single kink $x \mapsto \tanh((x - x_0)/\sqrt{2})$.
- If u_0 is in $X_{\text{bist}}(+1, -1)$, then there exists $x_0 \in \mathbb{R}$ such that $S_t u_0$ approaches the single kink $x \mapsto \tanh((x_0 - x)/\sqrt{2})$.

This result is implicit in many papers since this Allen–Cahn model is the simplest exhibiting this kind of long-range interaction, and consequently has been the most studied. A reference from which it directly follows is Ei’s paper [7] (where other references can be found).

10.2 Over-damped sine–Gordon equation

The equation reads (see example (b) of figure 2):

$$u_t = -\sin u + u_{xx} = -V'(u) + u_{xx} \quad \text{where} \quad V(u) = -\cos u + 1.$$

In this example the set \mathcal{M}_0 is $2\pi\mathbb{Z}$. Stationary solutions connecting equilibria in this set are: a “kink” connecting 0 to 2π , an “antikink” connecting 2π to 0 , their translates with respect to x , and their $2\pi\mathbb{Z}$ -translates with respect to u .

According to the maximum principle, for every pair (q_-, q_+) in \mathbb{Z}^2 and every initial condition u_0 in $X_{\text{bist}}(2\pi q_-, 2\pi q_+)$, the corresponding solution is bounded, and therefore the conclusions of Theorem 2 hold (the potential can be changed without changing the solution in order hypothesis (H_{coerc}) to be satisfied). According to these conclusions, the solution converges, when $t \rightarrow +\infty$, towards a standing terrace involving a finite number of kinks and antikinks, getting slowly away from one another.

Again, since the long-range interaction between two consecutive kink and antikink solutions is attractive, this standing terrace actually involves either $q_+ - q_-$ kinks (if q_+ is larger than q_-), or $q_- - q_+$ antikinks (if q_- is larger than q_+), or is reduced to the homogeneous equilibrium q_+ if q_+ and q_- are equal. Again, this follows from the results stated by Ei in [7].

10.3 Nagumo equation

The equation reads (see example (c) of figure 2):

$$u_t = -u(u-a)(u-1) + u_{xx} = -V'(u) + u_{xx}$$

where

$$V(u) = a\frac{u^2}{2} - (a+1)\frac{u^3}{3} + \frac{u^4}{4} \quad \text{and} \quad 0 < a < 1/2.$$

In this case the set \mathcal{M}_0 is reduced to the (local) minimum point 0, the bistable potential V reaches its global minimum at 1 (thus $V(1)$ is negative), and the set $\mathcal{S}_{\text{bist}}(\mathcal{M}_0)$ is reduced to a single stationary solution h homoclinic to 0 (and its translates with respect to x). As is well known this solution has one dimension of instability.

According to Corollary 6 on page 51, the set $\partial\mathcal{B}_{\text{att}}(0) \cap X_{\text{bist}}(0,0)$ is non empty, and, for every initial condition u_0 in this set, the asymptotic energy $\mathcal{E}_\infty[u_0]$ is positive. Thus, the conclusions of Theorem 2 hold for this initial condition: the corresponding solution $S_t(u_0)$ approaches a standing terrace involving a finite (nonzero) number of translates of h , getting slowly away from one another, when t approaches $+\infty$.

Once again, the long-range interaction between two consecutive translates of h is attractive (Ei [7]), therefore there should actually be only one translate of h in the standing terrace. Thus, there should exist $x_0 \in \mathbb{R}$ such that this solution $S_t(u_0)$ approaches the translate $x \mapsto h(x - x_0)$ of h , uniformly with respect to x , when t approaches $+\infty$. To my knowledge a rigorous proof of this claim is still missing, since it would require a statement analogous to the ones provided by Ei in [7], but in a slightly more general setting where the localized pulses or fronts are not necessarily stable, but may display a finite number of unstable modes. Indeed, in this example the stationary solution h has one unstable mode (its stable manifold is the border of the basin of attraction of the “metastable” homogeneous equilibrium 0 — this has been stated by many authors for a long time, see for instance [11, 25]).

Similar conclusions can be drawn about the *over-damped sine-Gordon equation with constant forcing* (see example (d) of figure 2):

$$u_t = -\sin u + \Omega + u_{xx} \quad \text{with} \quad 0 < \Omega < 1.$$

10.4 “Subcritical” Allen–Cahn equation

The equation reads (see example (e) of figure 2):

$$u_t = -u + u^3 + u_{xx} = -V'(u) + u_{xx} \quad \text{where} \quad V(u) = \frac{u^2}{2} - \frac{u^4}{4} + \varepsilon \frac{u^6}{6},$$

and where ε is a small positive real quantity, the last term of the potential being there just to ensure coercivity. In this example the set \mathcal{M}_0 is reduced to the (local) minimum point $\{0\}$, and the set $\mathcal{S}_{\text{bist}}(\mathcal{M}_0)$ is made of two stationary solutions homoclinic to 0, say h_+ (taking positive values) and h_- (taking negative values), and their translates with respect to x .

For every initial condition u_0 in $\partial\mathcal{B}_{\text{att}}(0) \cap X_{\text{bist}}(0,0)$ such that the corresponding solution is bounded (uniformly in x and t), the asymptotic energy $\mathcal{E}_\infty[u_0]$ is positive and the conclusions of Theorem 2 hold, that is the solution converges towards a standing terrace involving a finite (nonzero) number of translates of h_+ and h_- , getting slowly away from one another.

Once more, the long-range interaction between two consecutive translates of h_+ or two consecutive translates of h_- is attractive (Ei [7]), and therefore, such two consecutive translates of the same stationary solution should not take place in the asymptotic terrace. But again in this case, a rigorous proof of this claim is to my knowledge still missing since each of the stationary solutions h_+ and h_- has one unstable mode.

11 Attracting ball for the semi-flow

The aim of this section is to prove the existence of an attracting ball in X for the semi-flow of system (1) (Lemma 1 on page 20). This section presents strong similarities with appendix A.1 of the previous work [26] and especially section 2 of Gallay and Joly's paper [12], although the hypotheses and presentation are slightly different.

Since hypotheses (H_{norm}) , (H_{min}) , and (H_{disc}) have nothing to do with this result, it is sufficient here to assume that the potential V satisfies hypothesis (H_{coerc}) only. According to this hypothesis, there exist positive quantities q_{coerc} and K_{coerc} such that, for all u in \mathbb{R}^n ,

$$u \cdot \nabla V(u) \geq q_{\text{coerc}} u^2 - K_{\text{coerc}}.$$

First let us make an observation, besides of the proof itself: with the notation of subsection 3.3, expression (12) on page 21 (time derivative of a localized L^2 functional) yields, for a generic (nonnegative) weight function ψ in $W^{2,1}(\mathbb{R}, \mathbb{R}_+)$,

$$(62) \quad \frac{d}{dt} \int_{\mathbb{R}} \psi \frac{u^2}{2} dx \leq \int_{\mathbb{R}} \left[\psi (-q_{\text{coerc}} u^2 + K_{\text{coerc}}) + \psi'' \frac{|u|_{\mathcal{D}}^2}{2} \right] dx.$$

Thus, if the weight function ψ is such that $\lambda_{\mathcal{D}, \max} \psi''$ is not larger than $q_{\text{coerc}} \psi$, for instance:

$$\psi(x) = \exp\left(-\sqrt{\frac{q_{\text{coerc}}}{\lambda_{\mathcal{D}, \max}}}(x - x_0)\right),$$

then inequality (62) above yields

$$\frac{d}{dt} \int_{\mathbb{R}} \psi \frac{u^2}{2} dx \leq -\frac{q_{\text{coerc}}}{2} \int_{\mathbb{R}} \psi u^2 dx + K_{\text{coerc}} \int_{\mathbb{R}} \psi dx.$$

from which follows the existence of an attracting ball in the uniformly local Sobolev space $L^2_{\text{ul}}(\mathbb{R})$. The proof of the existence of an attracting ball in X will by contrast require a combination of both localized energy and L^2 -norm.

Hypothesis (H_{coerc}) guarantees that V is bounded from below on \mathbb{R}^n ; let us write, for all u in \mathbb{R}^n ,

$$V_0(u) = V(u) - \min_{v \in \mathbb{R}^n} V(v); \quad \text{thus,} \quad \min_{u \in \mathbb{R}^n} V_0(u) = 0.$$

Take u_0 in X and let

$$u : \mathbb{R}^n \times [0, T_{\max}), \quad (x, t) \mapsto u(x, t) = (S_t u_0)(x)$$

denote the (maximal) solution of system (1) with initial data u_0 , where T_{\max} in $(0, +\infty]$ denotes the upper bound of the (maximal) time interval where this solution is defined.

We are going to define a quantity κ_0 and functions ψ_0 and \mathcal{F}_0 that will play similar roles as the quantity κ and the functions ψ and \mathcal{F} that were defined in subsection 4.2. Since the definitions slightly differ, the subscript “0” is added to avoid confusion and to recall that these new objects are related to the “normalized” potential V_0 .

Let κ_0 be a positive quantity, small enough so that

$$\kappa_0^2 \frac{\lambda_{\mathcal{D}, \max}}{2} \leq \frac{q_{\text{coerc}}}{2} \quad \text{and} \quad \kappa_0^2 \lambda_{\mathcal{D}, \max} \leq 2 \quad (\text{namely: } \kappa_0 = \sqrt{\frac{\min(2, q_{\text{coerc}})}{\lambda_{\mathcal{D}, \max}}})$$

(those are the conditions that yield inequality (63) below) and let us consider the weight function ψ_0 defined (as in subsection 4.2 on page 24) by:

$$\psi_0(x) = \exp(-\kappa_0 |x|).$$

Finally, for all t in $[0, T_{\max})$ and ξ in \mathbb{R} , let

$$\begin{aligned} \mathcal{F}_0(\xi, t) &= \int_{\mathbb{R}} T_{\xi} \psi_0(x) \left(\frac{|u_x(x, t)|_{\mathcal{D}}^2}{2} + V_0(u(x, t)) + \frac{u(x, t)^2}{2} \right) dx, \\ \mathcal{Q}(\xi, t) &= \int_{\mathbb{R}} T_{\xi} \psi_0(x) \left(\frac{|u_x(x, t)|_{\mathcal{D}}^2}{2} + \frac{u(x, t)^2}{2} \right) dx, \end{aligned}$$

where $T_{\xi} \psi_0(x)$ is defined as in subsection 4.2. Obviously, the definition of V_0 ensures that:

$$\mathcal{F}_0(\xi, t) \geq \mathcal{Q}(\xi, t).$$

According to the generic expressions (11) and (12) of subsection 3.3, the functional \mathcal{F}_0 is expected to decrease with time, at least — because of the coercivity hypothesis (H_{coerc}) — where $u(x, t)$ is large; this decrease will be used to control the functional \mathcal{Q} . According to expressions (11) and (12) on page 21 (time derivatives of localized energy and L^2

functionals), for all t in $[0, T_{\max})$ and ξ in \mathbb{R} ,

$$\begin{aligned} \partial_t \mathcal{F}_0(\xi, t) &\leq \int_{\mathbb{R}} T_{\xi} \psi_0(x) \left(-u_t^2 + \kappa_0 |\mathcal{D}u_x \cdot u_t| - q_{\text{coerc}} u^2 + K_{\text{coerc}} - |u_x|_{\mathcal{D}}^2 + \frac{\kappa_0^2}{2} |u|_{\mathcal{D}}^2 \right) dx \\ &\leq K_{\text{coerc}} \int_{\mathbb{R}} \psi_0(x) dx + \int_{\mathbb{R}} T_{\xi} \psi_0(x) \left(-\frac{1}{2} q_{\text{coerc}} u^2 - \frac{1}{2} |u_x|_{\mathcal{D}}^2 \right) dx \\ &\quad + \int_{\mathbb{R}} T_{\xi} \psi_0(x) \left(-u_t^2 + \kappa_0 |\mathcal{D}u_x \cdot u_t| - \frac{1}{2} |u_x|_{\mathcal{D}}^2 \right) dx \\ &\quad + \int_{\mathbb{R}} T_{\xi} \psi_0(x) \left(-\frac{q_{\text{coerc}}}{2} u^2 + \frac{\kappa_0^2}{2} |u|_{\mathcal{D}}^2 \right) dx. \end{aligned}$$

According to the choice of κ_0 , the two last integrals are negative, thus

$$(63) \quad \partial_t \mathcal{F}_0(\xi, t) \leq -\min(q_{\text{coerc}}, 1) \mathcal{Q}(\xi, t) + \frac{2K_{\text{coerc}}}{\kappa_0}.$$

Let us consider the positive quantity

$$Q_{\mathcal{F}-\text{decr}} = \frac{1}{\min(q_{\text{coerc}}, 1)} \left(1 + \frac{2K_{\text{coerc}}}{\kappa_0} \right).$$

It follows from inequality (63) above that, for all t in $[0, T_{\max})$ and ξ in \mathbb{R} ,

$$\mathcal{Q}(\xi, t) \geq Q_{\mathcal{F}-\text{decr}} \implies \partial_t \mathcal{F}_0(\xi, t) \leq -1.$$

There is a last small difficulty to overcome, since the functional on the left-hand side of this implication is $\mathcal{Q}(\xi, t)$ — of course it would be even better if it was $\mathcal{F}_0(\xi, t)$. And unfortunately, the fact that the quantity $\mathcal{F}_0(\xi, t)$ is large does not automatically ensure that $\mathcal{Q}(\xi, t)$ itself is large; indeed the reason why $\mathcal{F}_0(\xi, t)$ is large could be that the term $V(u(x, t))$ takes very large values (much more than $|u(x, t)|^2$) far away in space from ξ , thus far from the bulk of the weight function $T_{\xi} \psi_0$ (see figure 15). In this case, the term $|u(x, t)|^2$ in $\mathcal{Q}(\xi, t)$ could count for nothing if it takes large values only far away from ξ .

Hopefully, this description of the enemy furnishes by the way the weapon: if $\mathcal{F}_0(\xi, t)$ is very large while $\mathcal{Q}(\xi, t)$ remains below the quantity $Q_{\mathcal{F}-\text{decr}}$, this probably means that $\mathcal{F}_0(\xi, t)$ is (much) smaller than its supremum over all possible values of ξ . As a consequence, if $\mathcal{F}_0(\xi, t)$ is large *and* close to its supremum, then the inconvenience above should not occur and $\mathcal{Q}(\xi, t)$ should be large, and thus $\partial_t \mathcal{F}_0(\xi, t)$ should be negative. These considerations are formalized by the next lemma.

For t in $[0, T_{\max})$ let

$$\overline{\mathcal{F}}_0(t) = \sup_{\xi \in \mathbb{R}} \mathcal{F}_0(\xi, t)$$

(since the function $x \mapsto u(x, t)$ is in X , this quantity is finite).

Lemma 18 (\mathcal{Q} small and \mathcal{F}_0 large means supremum of \mathcal{F}_0 attained elsewhere). *There exists a positive quantity $F_{\text{sup-higher}}$, depending (only) on V and \mathcal{D} , such that, for all ξ in \mathbb{R} and t in $[0, T_{\max})$,*

$$\left(\mathcal{Q}(\xi, t) \leq Q_{\mathcal{F}-\text{decr}} \quad \text{and} \quad \mathcal{F}_0(\xi, t) \geq F_{\text{sup-higher}} \right) \implies \overline{\mathcal{F}}_0(t) \geq \mathcal{F}_0(\xi, t) + 1.$$

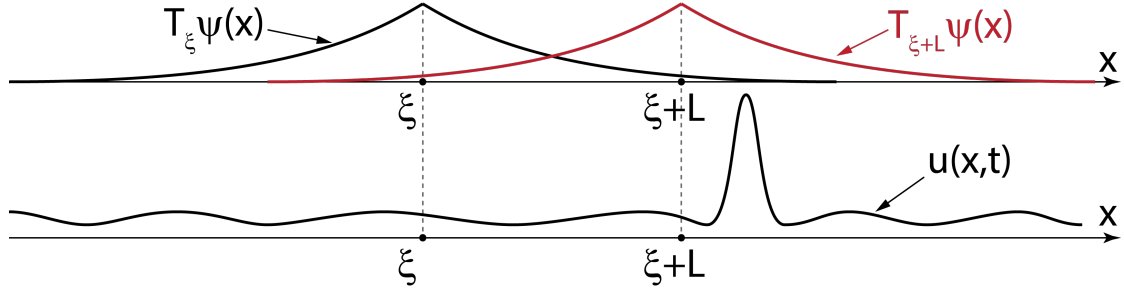


Figure 15: Illustration of Lemma 18. If the quantity $\mathcal{F}_0(\xi, t)$ is very large whereas the quantity $\mathcal{Q}(\xi, t)$ is not, this means there must be a high contribution of the potential term due to a large excursion of $u(x, t)$ far from ξ (to the right of ξ on the figure), and as a consequence $\mathcal{F}_0(\cdot, t)$ reaches a higher value at $\xi + L$ than at ξ .

This lemma is illustrated by figure 15.

Proof of Lemma 18. Let L be a positive quantity, large enough so that

$$\exp(-\kappa_0 L) \leq \frac{1}{3}, \quad \text{namely} \quad L = \frac{\log(3)}{\kappa_0}.$$

There exists a quantity F_{loc} , depending (only) on V and \mathcal{D} , such that, for all ξ in \mathbb{R} and t in $[0, T_{\text{max}})$,

$$\mathcal{Q}(\xi, t) \leq Q_{\mathcal{F}\text{-decr}} \Rightarrow \int_{\xi-L}^{\xi+L} T_\xi \psi_0(x) \left(\frac{|u_x(x, t)|_{\mathcal{D}}^2}{2} + V_0(u(x, t)) + \frac{u(x, t)^2}{2} \right) dx \leq F_{\text{loc}}.$$

Thus, if $\mathcal{Q}(\xi, t) \leq Q_{\mathcal{F}\text{-decr}}$, then according to the definition of \mathcal{F}_0 at least one of the following inequalities holds:

(64)

$$\begin{aligned} \text{either} \quad & \int_{-\infty}^{\xi-L} T_\xi \psi_0(x) \left(\frac{|u_x(x, t)|_{\mathcal{D}}^2}{2} + V_0(u(x, t)) + \frac{u(x, t)^2}{2} \right) dx \geq \frac{1}{2} (\mathcal{F}_0(\xi, t) - F_{\text{loc}}), \\ \text{or} \quad & \int_{\xi+L}^{+\infty} T_\xi \psi_0(x) \left(\frac{|u_x(x, t)|_{\mathcal{D}}^2}{2} + V_0(u(x, t)) + \frac{u(x, t)^2}{2} \right) dx \geq \frac{1}{2} (\mathcal{F}_0(\xi, t) - F_{\text{loc}}). \end{aligned}$$

Take and fix ξ in \mathbb{R} and t in $[0, T_{\text{max}})$ such that $\mathcal{Q}(\xi, t) \leq Q_{\mathcal{F}\text{-decr}}$, and assume for instance that the first of the two inequalities (64) above holds. Observe moreover that, according to the choice of L , for all x in $(-\infty, \xi - L]$,

$$T_{\xi-L} \psi_0(x) = \exp(\kappa_0 L) T_\xi \psi_0(x) \geq 3 T_\xi \psi_0(x),$$

thus, since the integrand in $\mathcal{F}_0(\cdot, \cdot)$ is nonnegative, the first of the two inequalities (64) above yields:

$$\mathcal{F}_0(\xi - L, t) \geq \frac{3}{2} (\mathcal{F}_0(\xi, t) - F_{\text{loc}}),$$

or equivalently

$$\mathcal{F}_0(\xi - L, t) \geq \mathcal{F}_0(\xi, t) + \frac{1}{2}(\mathcal{F}_0(\xi, t) - 3F_{\text{loc}}),$$

and this shows that the lemma holds for the following choice of $F_{\text{sup-higher}}$:

$$F_{\text{sup-higher}} = 3F_{\text{loc}} + 2.$$

□

It follows from Lemma 18 that, for all t in $[0, T_{\text{max}})$,

$$\overline{\mathcal{F}}_0(t) \leq \max(F_{\text{sup-higher}}, \overline{\mathcal{F}}_0(0) - t),$$

thus

$$\sup_{\xi \in \mathbb{R}} \mathcal{Q}(\xi, t) \leq \max(F_{\text{sup-higher}}, \overline{\mathcal{F}}_0(0) - t)$$

and these estimates hold whatever the initial data u_0 in X . In view of the definition of \mathcal{Q} , the last inequality shows that the semi-flow is globally defined and admits an attracting ball in the Sobolev space $H_{\text{ul}}^1(\mathbb{R}, \mathbb{R}^n)$, and the conclusions of Lemma 1 follow.

12 Properties of solutions of the Hamiltonian system governing stationary solutions

This section is devoted to the proof of some properties of solutions of the Hamiltonian system (4) on page 7 governing stationary solutions of system (1). Recall that this Hamiltonian system reads:

$$(65) \quad \mathcal{D} \frac{d^2 u}{dx^2} = \nabla V(u) \quad \text{or} \quad \frac{d}{x} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \mathcal{D}^{-1} \nabla V(u) \end{pmatrix} = \Omega \cdot \nabla H(u, v)$$

where $x \mapsto u(x)$ and $x \mapsto v(x)$ are functions taking their values in \mathbb{R}^n , and

$$(66) \quad H : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}, \quad (u, v) \mapsto \frac{|v|_{\mathcal{D}}^2}{2} - V(u) \quad \text{and} \quad \Omega = \begin{pmatrix} 0 & \mathcal{D}^{-1} \\ -\mathcal{D}^{-1} & 0 \end{pmatrix}.$$

Let us assume in this section (thus in the two next subsections) that V satisfies hypotheses (H_{coerc}) , (H_{norm}) , and (H_{min}) . Recall that the parameter d_{Esc} was defined in section 2 (see (8) in subsection 2.11).

12.1 Stationary solutions remaining in a neighbourhood of a minimum point approach this point

Let

$$u : \mathbb{R} \rightarrow \mathbb{R}^n, \quad x \mapsto u(x)$$

denote a solution of system (65) above, defined on \mathbb{R} (some solutions of this system may blow up in finite time, but those are not considered here), and let m be a minimum point

of V in the level set $V^{-1}(\{0\})$. Let us consider the the (open/closed) ball of center m and radius d_{Esc} for the $|\cdot|_{\mathcal{D}}$ -norm:

$$\mathcal{B}(m, d_{\text{Esc}}) = \{w \in \mathbb{R}^n : |w - m|_{\mathcal{D}} < d_{\text{Esc}}\}$$

$$\text{and } \overline{\mathcal{B}(m, d_{\text{Esc}})} = \{w \in \mathbb{R}^n : |w - m|_{\mathcal{D}} \leq d_{\text{Esc}}\}.$$

Our aim in this subsection is to prove the following lemma, which is a direct consequence of the fact that the equilibrium $(m, 0)$ is hyperbolic for the Hamiltonian system (65).

Lemma 19 (spatial asymptotics of stationary solutions). *The following assertions hold.*

1. *If $u(x)$ is in $\overline{\mathcal{B}(m, d_{\text{Esc}})}$ for all x in \mathbb{R} , then $u(\cdot)$ is identically equal to m .*
2. *If $u(x)$ is in $\overline{\mathcal{B}(m, d_{\text{Esc}})}$ for all x in \mathbb{R}_+ , then:*
 - *$u(x) \rightarrow m$ and $u'(x) \rightarrow 0$ when $x \rightarrow +\infty$,*
 - *and $u(x)$ is actually in $\mathcal{B}(m, d_{\text{Esc}})$ for all x in \mathbb{R}_+^* ;*
 - *if moreover $u(\cdot)$ is not identically equal to m , then*

$$\langle u(x) - m, u'(x) \rangle_{\mathcal{D}} < 0 \quad \text{for all } x \text{ in } \mathbb{R}_+.$$

Proof. Let us define the function $Q : \mathbb{R} \rightarrow \mathbb{R}$ by:

$$Q(x) = \frac{|u(x) - m|_{\mathcal{D}}^2}{2}.$$

For all x in \mathbb{R} ,

$$Q'(x) = \langle u(x) - m, u'(x) \rangle_{\mathcal{D}} \quad \text{and} \quad Q''(x) = |u'(x)|_{\mathcal{D}}^2 + (u(x) - m) \cdot \nabla V(u(x)).$$

Let us assume that $u(x)$ belongs to $\overline{\mathcal{B}(m, d_{\text{Esc}})}$ for all x in \mathbb{R}_+ . According to properties (13) on page 22 following the definition of d_{Esc} , this yields, for all x in \mathbb{R}_+ ,

$$(67) \quad Q''(x) \geq |u'(x)|_{\mathcal{D}}^2 + \frac{\lambda_{V, \min}}{2} (u(x) - m)^2 \geq 0.$$

Since the Hamiltonian

$$\frac{|u'(x)|_{\mathcal{D}}^2}{2} - V(u(x))$$

is constant and since $u(\cdot)$ is bounded on \mathbb{R}_+ , it follows that $u'(\cdot)$ is also bounded on \mathbb{R}_+ . Thus the function $Q'(\cdot)$ is bounded on \mathbb{R}_+ , and thus according to the lower bound (67) on $Q''(\cdot)$ it must approach a finite limit when x approaches $+\infty$. In other words, the function $Q''(\cdot)$ is integrable on \mathbb{R}_+ .

Besides, according to the Hamiltonian system (65), the function $u''(\cdot)$ is bounded on \mathbb{R}_+ , thus the same is true for the function $Q'''(\cdot)$. As a consequence, the quantity $Q''(x)$ must approach 0 when x approaches $+\infty$. It follows from the lower bound (67) on $Q''(\cdot)$ that

$$u(x) \rightarrow m \quad \text{and} \quad u'(x) \rightarrow 0 \quad \text{when } x \rightarrow +\infty.$$

Thus $Q'(x) \rightarrow 0$ when $x \rightarrow +\infty$ and therefore $Q'(\cdot)$ is nonpositive on \mathbb{R}_+ .

If in addition $u(\cdot)$ is not identically equal to m , then it follows from the lower bound (67) on $Q''(\cdot)$ that the quantity $Q''(\cdot)$ is actually positive for all \mathbb{R}_+ . As a consequence, the function $Q'(\cdot)$ is strictly decreasing on \mathbb{R}_+ . Thus, for all x in \mathbb{R}_+^* ,

$$Q(x) < Q(0) \quad \text{thus} \quad |u(x) - m|_{\mathcal{D}} < d_{\text{Esc}},$$

which proves the first assertion.

It remains to consider the case where $u(x)$ belongs to $\overline{\mathcal{B}(m, d_{\text{Esc}})}$ for all x in \mathbb{R} . In this case, the same arguments show that the quantity $Q''(x)$ is at the same time nonnegative and nonpositive (and thus equal to 0) for all x in \mathbb{R} . It follows from the lower bound (67) on $Q''(\cdot)$ that $u(\cdot)$ is identically equal to m , and this completes the proof. \square

12.2 Lagrangian of stationary solutions with almost zero Hamiltonian

Notation. Let us consider the ‘‘Lagrangian’’ function

$$L : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}, \quad (u, v) \mapsto \frac{|v|_{\mathcal{D}}^2}{2} + V(u).$$

Definition. If $x \mapsto u(x)$ is a solution of the Hamiltonian system (65) that is defined for all x in \mathbb{R} (in other words, for which no blow-up occurs), let us call *Lagrangian* of this solution the (finite or infinite) quantity:

$$\mathcal{L}[x \mapsto u(x)] = \int_{\mathbb{R}} L(u(x), u'(x)) \, dx,$$

provided that this integral can be unambiguously defined, that is: provided that the integral is convergent, or that it diverges to $+\infty$ at both ends of \mathbb{R} , or that it diverges to $-\infty$ at both ends of \mathbb{R} .

The aim of this subsection is to prove the following proposition. Recall that $\mathcal{S}_{\text{bist}}(\mathcal{M}_0)$ denotes the set of solutions $x \mapsto u(x)$ of the Hamiltonian system (65) that are homoclinic or heteroclinic to minimum points of V in the level set $V^{-1}(\{0\})$.

Proposition 4 (almost zero Hamiltonian and finite Lagrangian means bistable). *There exists a positive quantity δ_{Ham} such that, for every solution of the Hamiltonian system (65) that is defined on the whole real line, if*

- *the Hamiltonian of this solution is between $-\delta_{\text{Ham}}$ and $+\delta_{\text{Ham}}$,*
- *and this solution does not belong to the set $\mathcal{S}_{\text{bist}}(\mathcal{M}_0)$,*

then the Lagrangian of this solution is equal to plus infinity.

Hypothesis (H_{\min}) (namely the fact that every critical point in the level set $V^{-1}(\{0\})$ is a nondegenerate minimum point) plays an essential role in the proof of this proposition. By the way, the proposition is false without this hypothesis.

Proof. If $x \mapsto u(x)$ is a solution of the Hamiltonian system (65) that is defined for all x in \mathbb{R} , let

$$\Sigma_{\text{Esc}}[x \mapsto u(x)] = \Sigma_{\text{Esc}}[u(\cdot)] = \{x \in \mathbb{R} : \text{for all } m \text{ in } \mathcal{M}_0, |u(x) - m|_{\mathcal{D}} > d_{\text{Esc}}\}$$

(observe the analogy with the notation $\Sigma_{\text{Esc}}(t)$ in subsection 4.2).

It follows from properties (13) on page 22 following the definition (8) of d_{Esc} that, if $x \mapsto u(x)$ is a solution of the Hamiltonian system (65) that is defined for all x in \mathbb{R} , then

$$(68) \quad L(u(x), u'(x)) \geq 0 \quad \text{for all } x \text{ in } \mathbb{R} \setminus \Sigma_{\text{Esc}}[u(\cdot)].$$

The proof will follow from the next two lemmas.

Lemma 20 (non bistable solutions never stop to “Escape”). *For every solution $x \mapsto u(x)$ of the Hamiltonian system (65) that is defined for all x in \mathbb{R} and that is not in $\mathcal{S}_{\text{bist}}(\mathcal{M}_0)$, the set $\Sigma_{\text{Esc}}[u(\cdot)]$ is unbounded.*

Proof of Lemma 20. This lemma is an immediate consequence of Lemma 19 of the previous subsection 12.1. \square

Lemma 21 (almost zero Hamiltonian yields positive Lagrangian at each “Escape”). *There exist positive quantities δ_{Ham} and δ_{Lag} such that, for every solution $x \mapsto u(x)$ of the Hamiltonian system (65) that is defined for all x in \mathbb{R} , if the Hamiltonian of this solution is between $-\delta_{\text{Ham}}$ and $+\delta_{\text{Ham}}$, then, for every x_0 in \mathbb{R} , the following holds:*

$$[x_0, x_0 + 1] \cap \Sigma_{\text{Esc}}[u(\cdot)] \neq \emptyset \implies \int_{x_0}^{x_0+1} L(u(x), u'(x)) dx \geq \delta_{\text{Lag}}.$$

Proof of Lemma 21. Let us proceed by contradiction and assume that, for every integer p , there exists a solution $x \mapsto u_p(x)$ of the Hamiltonian system (65) that is defined for all x in \mathbb{R} , such that the Hamiltonian of this solution is between $-1/p$ and $+1/p$, and such that there exists x_p in \mathbb{R} such that

$$[x_p, x_{p+1}] \cap \Sigma_{\text{Esc}}[u_p(\cdot)] \neq \emptyset \quad \text{and} \quad \int_{x_p}^{x_{p+1}} L(u_p(x), u'_p(x)) dx \leq \frac{1}{p}.$$

A compactness argument will lead to the sought contradiction.

For notational convenience, let us assume without loss of generality (up to replacing $x \mapsto u_p(x)$ by $x \mapsto u_p(x - x_p)$) that x_p equals 0. It follows from this estimate and from the fact that the Hamiltonian of the solution is between $-1/p$ and $+1/p$ that:

$$\int_0^1 |u'_p(x)|_{\mathcal{D}}^2 dx \leq \frac{2}{p} \quad \text{and} \quad \int_0^1 V(u_p(x)) dx \leq \frac{2}{p}.$$

According to the first of these inequalities, $u_p(\cdot)$ varies by less than $1/\sqrt{p}$ on $[0, 1]$, and according to the second inequality $u_p(0)$ is bounded independently of n (indeed according to the coercivity hypothesis (H_{coerc}) , the quantity $V(v)$ approaches plus infinity as $|v|$ approaches plus infinity).

Thus, up to extracting a subsequence, we may assume that the sequence of maps $x \mapsto u_p(x)$ converges, uniformly on $[0, 1]$, towards an equilibrium u_∞ of the Hamiltonian system (65) satisfying:

$$V(u_\infty) = 0 \quad \text{and} \quad |u_\infty - m|_{\mathcal{D}} \geq d_{\text{Esc}} \text{ for all } m \text{ in } \mathcal{M}_0,$$

a contradiction with the definition of \mathcal{M}_0 and hypothesis (H_{\min}) . \square

We are now in position to complete the proof of Proposition 4. Let $x \mapsto u(x)$ be a solution the Hamiltonian system (65) that is defined for all x in \mathbb{R} , and such that:

1. the Hamiltonian of this solution is between $-\delta_{\text{Ham}}$ and $+\delta_{\text{Ham}}$,
2. and this solution is in $\mathcal{S}_{\text{bist}}(\mathcal{M}_0)$.

Then, for every positive quantity x (say larger than 1),

$$\int_0^x L(u(y), u'(y)) dy = \int_0^{\text{frac}(x)} L(u(y), u'(y)) dy + \sum_{i=0}^{\text{int}(x)-1} \int_{\text{frac}(x)+i}^{\text{frac}(x)+i+1} L(u(y), u'(y)) dy$$

and the i -th term under the sum of the right-hand side of this equality is:

- nonnegative if the intersection

$$[\text{frac}(x) + i, \text{frac}(x) + i + 1] \cap \Sigma_{\text{Esc}}[u(\cdot)]$$

is empty (according to assertion (68)),

- not smaller than δ_{Lag} if this intersection is nonempty (according to Lemma 21 about the non-negativity of $L(u(\cdot), u'(\cdot))$),

and according to Lemma 20 the second of these two alternatives occurs for an unbounded number of values of i when x grows to plus infinity. As a consequence (applying the symmetric argument at the left of 0), both quantities

$$\int_0^x L(u(y), u'(y)) dy \quad \text{and} \quad \int_{-x}^0 L(u(y), u'(y)) dy$$

approach plus infinity as x approaches plus infinity. Proposition 4 is proved. \square

13 The space of asymptotic patterns

The aim of this section is to make a few (rather abstract) remarks concerning the regularity (more precisely, the upper semi-continuity) of the correspondence between an initial condition and the distribution of energy in the standing terrace provided by Theorem 2 on page 15 when the asymptotic energy of the corresponding solution is not $-\infty$.

Let us assume that the hypotheses of Theorem 2 are satisfied. Let us consider the space:

$$X_{\text{bist, no-inv}}(\mathcal{M}_0) = X_{\text{bist}}(\mathcal{M}_0) \cap \mathcal{E}_{\infty}^{-1}([0, +\infty)) ,$$

and, for every pair (m_-, m_+) of points of \mathcal{M}_0 ,

$$X_{\text{bist, no-inv}}(m_-, m_+) = X_{\text{bist}}(m_-, m_+) \cap \mathcal{E}_{\infty}^{-1}([0, +\infty)) .$$

In this notation, the additional subscript “no-inv” refers to the fact, that, for those initial data, the stable equilibria at both ends of space are not “invaded” by travelling fronts. Indeed, Proposition 7 of [27] states (under the additional hypothesis that the diffusion matrix \mathcal{D} is the identity matrix) that solutions in $X_{\text{bist}}(\mathcal{M}_0)$ having an asymptotic energy equal to $-\infty$ are exactly those for which the equilibria at both ends of space are invaded by bistable travelling fronts.

For every u_0 in $X_{\text{bist, no-inv}}(\mathcal{M}_0)$, let us denote by $q_{\infty}[u_0]$ the integer q defined by the conclusions of Theorem 2 (the “number of items in the standing terrace”). This defines a map:

$$(69) \quad q_{\infty} : X_{\text{bist, no-inv}}(\mathcal{M}_0) \rightarrow \mathbb{N} .$$

Obviously, for every local minimum m in \mathcal{M}_0 ,

$$\mathcal{B}_{\text{att}}(m) = X_{\text{bist, no-inv}}(m, m) \cap q_{\infty}^{-1}(\{0\})$$

(this statement has no interest in itself, it is just written here to get familiar with the notation). The following proposition is an obvious consequence of Corollary 6 on page 51.

Proposition 5 (the number of items in the standing terrace is not lower semi-continuous with respect to the initial condition). *Assume that V satisfies the hypotheses of Theorem 2 and assume in addition that V takes negative values. Then the number of items in the asymptotic standing terrace is not lower semi-continuous with respect to the initial condition. In more formal terms, the map $q_{\infty}[\cdot]$ defined in (69) is not lower semi-continuous*

Proof. Since V takes negative values, according to Corollary 6 on page 51, for every m in \mathcal{M}_0 , the set $\partial\mathcal{B}_{\text{att}}(m)$ is nonempty, and for every initial condition u_0 , in this set, the integer $q_{\infty}[u_0]$ is nonzero. On the other, by definition of the topological border, u_0 is arbitrarily close to initial conditions in $\mathcal{B}_{\text{att}}(m)$, and for those initial condition the integer $q_{\infty}[\cdot]$ vanishes. \square

It is likely that this map $q_{\infty}[\cdot]$ is *not* upper semi-continuous in general (thus neither lower nor upper semi-continuous, in general). It would be interesting however to build an explicit example of a potential V for which $q_{\infty}[\cdot]$ is not upper semi-continuous (say, for which an unstable pulse may split into two repulsive “smaller” pulses). The conclusion that can be drawn from this observation is that the definition (69) of the map $q_{\infty}[\cdot]$ is “irrelevant” (let us say: “bad”), in the sense that it does not ensure upper semi-continuity. By contrast, any “good” definition of an asymptotic feature of a solution should display

some form of upper semi-continuity. In this sense, the asymptotic energy defined in subsection 2.7 is a “good” feature.

Unfortunately, the following definitions will turn to be naively “bad”. Thus the sole interest of the next lines is to raise the question of what would be the “good” definitions to choose in place of these “bad” ones.

Let us consider the following spaces (“bad” space of asymptotic profiles and “bad” space of asymptotic energy distributions):

$$\mathcal{P}_{\text{bad}} = \mathbb{R}^n \cup \bigsqcup_{q \in \mathbb{N}^*} (C^k(\mathbb{R}, \mathbb{R}^n) \cap H^1(\mathbb{R}, \mathbb{R}^n))^q \quad \text{and} \quad \mathcal{E}_{\text{bad}} = \{0\} \cup \bigsqcup_{q \in \mathbb{N}^*} \mathbb{R}^q.$$

The conclusions of Theorem 2 lead us to define the following map, that sends an initial condition to the profiles of the standing terrace provided by the conclusions of Theorem 2 (let us denote by $u_{\infty,1}, \dots, u_{\infty,q_\infty[u_0]}$ these profiles if $q_\infty[u_0]$ is positive):

$$\mathcal{P}_\infty : X_{\text{bist, no-inv}}(\mathcal{M}_0) \rightarrow \mathcal{P}_{\text{bad}}, \quad u_0 \mapsto \begin{cases} m_+ & \text{if } q_\infty[u_0] = 0, \\ (u_{\infty,1}, \dots, u_{\infty,q_\infty[u_0]}) & \text{if } q_\infty[u_0] > 0, \end{cases}$$

and the following map, that sends an “asymptotic pattern” to the corresponding “distribution of asymptotic energies”:

$$\mathcal{E} : \mathcal{P}_{\text{bad}} \rightarrow \mathcal{E}_{\text{bad}}, \quad m_+ \mapsto 0, \quad (u_1, \dots, u_q) \mapsto (\mathcal{E}[u_1], \dots, \mathcal{E}[u_q]),$$

and the following map, that does nothing more than summing up the components of a “distribution of asymptotic energies”:

$$\Sigma : \mathcal{E}_{\text{bad}} \rightarrow [0, +\infty), \quad 0 \mapsto 0, \quad (E_1, \dots, E_q) \mapsto \sum_{i=1}^q E_i,$$

and the following map, that simply counts the number of items in the asymptotic pattern:

$$\text{card} : \mathcal{E}_{\text{bad}} \rightarrow \mathbb{N}, \quad 0 \mapsto 0, \quad (E_1, \dots, E_q) \mapsto q.$$

As already mentioned, it is likely that the map

$$q_\infty = \text{card} \circ \mathcal{E} \circ \mathcal{P}_\infty$$

is not upper semi-continuous, whereas by contrast Proposition 2 states that the map

$$\mathcal{E}_\infty = \Sigma \circ \mathcal{E} \circ \mathcal{P}_\infty$$

is upper semi-continuous.

Unfortunately, there is no hope that, with the definitions above, the map $\mathcal{E} \circ \mathcal{P}_\infty$ may display any kind of upper semi-continuity. The sole goodness of the spaces \mathcal{P}_{bad} and \mathcal{E}_{bad} is that they bear a partial order that is relevant (only in space dimension one) with respect to the phenomenon under consideration, but this is far from being sufficient to ensure the desired upper semi-continuity. The problem of finding proper definitions for

these two spaces so that the map $\mathcal{E} \circ \mathcal{P}_\infty$ (together with the map “counting the number of items in the standing terrace”) be upper semi-continuous is beyond the scope of this paper.

The results of [27] (global behaviour of all bistable solutions under generic assumptions on the potential) raise the same kind of questions about the topological structure of the asymptotic pattern of every bistable solutions (and not only those of the set $X_{\text{bist, no-inv}}(\mathcal{M}_0)$), including the travelling fronts involved in this asymptotic pattern and their speeds.

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